Refutation of the Boone-Rogers theorem

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Abstract: The Boone-Rogers theorem states that the uniform word problem for the class of all finitely presented groups with solvable word problem is unsolvable. We show that is *no*t tautologous. We further show that a universal, solvable word problem group is tautologous. The former forms a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·; \ Not And;
> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥, zero;
(%z>#z) N as non-contingency, Δ, ordinal 1; (%z<#z) C as contingency, ▽, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: en.wikipedia.org/wiki/Word_problem_for_groups [Note that this entry is not well written.]

Word problem for groups
The related but different **uniform word problem** for a class *K* of recursively presented groups is the algorithmic problem of deciding, given as input a presentation *P* for a group *G* in the class *K* and two words in the generators of *G*, whether the words represent the same element of *G*. Some authors require the class *K* to be definable by a recursively enumerable set of presentations.

Unsolvability of the uniform word problem
The criterion given above, for the solvability of the word problem in a single group, can be extended by a straightforward argument. This gives the following criterion for the uniform solvability of the word problem for a class of finitely presented groups:

> To solve the uniform word problem for a class K of groups, it is sufficient to find a recursive function f(P,w) that takes a finite presentation P for a group G and a word w in the generators of G, such that whenever G ∈ K:

f(P,w) = {0 if w ≠ 1 in G
{undefined/does not halt if w = 1 in G (1.1)

LET p, q, r, s: P, G (or H), f (or h_n), w.

(r&(p&s))=((((s=~(%s>#s))<q)> (s@s)) + (((s= (%s>#s))<q)>~(s@s))) ;
FFFF FFFF FFFF FT**F**T (1.2)

Remark 1.2: For Eq. 1.2 as rendered to prove the solution of the uniform word problem, the result should be a theorem of all T's. In fact, the result is *not* a theorem, and also *not* a contradiction, but something else.

Boone-Rogers Theorem: There is no uniform partial algorithm that solves the word problem in all finitely presented groups with solvable word problem.

In other words, the uniform word problem for the class of all finitely presented groups with solvable word problem is unsolvable.

Remark 1.3: Eq. 1.2 contradicts the Boone-Rogers theorem with restatement to refute it.

Proof that there is no universal solvable word problem group
If H has solvable word problem, then at least one of these homomorphisms must be an embedding. So given a word w in the generators of H: (3.1)

If w≠1 in H, $h_n(w) \neq 1$ in G for some h_n
If w=1 in H, $h_n(w)=1$ in G for all h_n

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%( ((~(s=(%s>#s))<q)>((q<r)>~((r&s)=(%s>#s))))+
   (( (s=(%s>#s))<q)>((q<r)>  ((r&s)=(%s>#s)))))=(q=q) ;
                                   TTTT TTTT TTTT TTTT          (3.2)
```

The function f clearly depends on the presentation P. Considering it to be a function of the two variables, a recursive function f(P,w) has been constructed that takes a finite presentation P for a group H and a word w in the generators of a group G, such that whenever G has soluble word problem: (4.0.1.1)

f(P,w) =	{0	if w ≠ 1 in H	
	{undefined/does not halt	if w = 1 in H	(4.0.2.1)

Remark 4.0.1.1: We write Eq. 4.0.1.1 as If 3.1 and 1.1, then 4.1 (4.1)

```
(%( ((~(s=(%s>#s))<q)>((q<r)>~((r&s)=(%s>#s))))+(( (s=(%s>#s))<q)>((q<r)>
((r&s)=(%s>#s)))))&((r&(p&s))=((((s=~(%s>#s))<q)> (s@s))+(((s=
(%s>#s))<q)>~(s@s)))))>((r&(p&s))=((((s=~(%s>#s))<q)> (s@s))+(((s=
(%s>#s))<q)>~(s@s)))) ;            TTTT TTTT TTTT TTTT          (4.2)
```

But this uniformly solves the word problem for the class of all finitely presented groups with solvable word problem, contradicting Boone-Rogers. This contradiction proves G cannot exist.

Remark 4.2: Eq. 4.2 is tautologous, proving G can exist and that there is a universal, solvable word problem group.