Refutation of Hamkins' theorem

Abstract: Hamkins' theorem claims "every countable model … of set theory embeds into its own constructible universe … x∈y ↔ j(x)∈j(y)", which is *not* tautologous, forming a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·; \ Not And;
> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) N as non-contingency, Δ, ordinal 1; (%z<#z) C as contingency, ▽, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: Fuchs, G.; Gitman, V.; Hamkins, J.D. (2018). Incomparable ω_1-like models of set theory. arxiv.org/pdf/1501.01022.pdf

Abstract. We show that the analogues of the Hamkins embedding theorems .. , proved for the countable models of set theory, do not hold when extended to the uncountable realm of ω_1-like models of set theory. Specifically, under the ◊ hypothesis and suitable consistency assumptions, we show that there is a family of 2^{ω_1} many ω_1-like models of ZFC, all with the same ordinals, that are pairwise incomparable under embeddability; there can be a transitive ω_1-like model of ZFC that does not embed into its own constructible universe; and there can be an ω_1-like model of PA whose structure of hereditarily finite sets is not universal for the ω_1-like models of set theory.

1. Introduction We should like to consider the question of whether the embedding theorems of Hamkins .. , recently proved for the countable models of set theory, might extend to the realm of uncountable models. … The question we consider here is, do the analogous results hold for uncountable models? Our answer is that they do not.

The Hamkins embedding theorems are expressed collectively in theorem 1 below. An *embedding* of one model ... of set theory into another … is simply a function … ; note by extensionality that every embedding is injective.

Although this is the usual model-theoretic embedding concept for relational structures, the reader should note that it is a considerably weaker embedding concept than commonly encountered in set theory, because this kind of embedding need not be elementary nor even Δ_0-elementary, although clearly every embedding as just defined is elementary at least for quantifier-free assertions. So we caution the reader not to assume a greater degree of elementarity beyond quantifier-free elementarity for the embeddings appearing in this paper, except where we explicitly remark on it.

Theorem 1 (Hamkins) ..

> (3) Consequently, every countable model … of set theory embeds into its own constructible universe … x∈y ↔ j(x)∈j(y) (1.3.1)
>
> LET x, y, j: p, q, r
>
> (p<q)=((r&p)<(r&q)) ; `T`**`F`**`TT TTTT T`**`F`**`TT TTTT` (1.3.2)
>
> **Remark 1.3.2:** Eq. 1.3.2 is *not* tautologous, thereby refuting a seminal conjecture of Hamlin's theorem 1.

One can begin to get an appreciation for the difference in embedding concepts by observing that ZFC proves that there is a nontrivial embedding j: V→V , namely, the embedding recursively defined as follows j(y) = j(x)|x∈y∪ {∅,y}.

We leave it as a fun exercise to verify that x∈y ↔ j(x)∈ j(y) for the embedding j defined by this recursion.[1] [1]See [Ham13]; but to give a hint here for the impatient, note that every j(y) is nonempty and also ∅ ∉ j(y); it follows that inside j(y) we may identify the pair {∅,y} j(y); it follows that j is injective and furthermore, the only way to have j(x)∈j(y) is from x∈y.