Refutation of Keisler's ultraproduct construction

Abstract: The ultraproduct construction of Keisler is based on definitions for proper filter in six equations and for ultrafilter in two equations. The definitions are *not* tautologous. This refutes the ultraproduct construction as "a uniform method of building models of first order theories which has applications in many areas of mathematics." Claims by other writers to extend Keisler's construction similarly fail, to form a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·; \ Not And;
> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥, zero;
(%z>#z) N as non-contingency, Δ, ordinal 1; (%z<#z) C as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: Keisler, H.J. (2010). The ultraproduct construction.
math.wisc.edu/~keisler/ultraproducts-web-final.pdf

1. Introduction

The ultraproduct construction is a uniform method of building models of first order theories which has applications in many areas of mathematics. It is attractive because it is algebraic in nature, but preserves all properties expressible in first order logic.

2. Ultraproducts and ultrapowers

We begin with the definition of an ultrafilter over an index set I. An ultrafilter over I can be defined as the collection of all sets of measure 1 with respect to a finitely additive measure μ : P(I) → {0, 1}. Here is an equivalent definition in more primitive terms.

Definition 2.1. Let I be a non-empty set. (2.1.0.1.1)

LET p, q, r, s: I, U, X, Y.

p=~(p@p) ; **F**T**F**T **F**T**F**T **F**T**F**T **F**T**F**T (2.1.0.1.2)

A **proper filter** U over I is a set of subsets of I such that: (2.1.0.2.1)

((p=~(p@p))>(q>p))>(p>(p>p)) ;
TTTT TTTT TTTT TTTT (2.1.0.2.2)

(i) U is closed under supersets; if X ∈ U and X ⊆ Y ⊆ I then Y ∈ U.
(2.1.i.1)

((r<q)&~(~(p<s)<r))>(s<q) ;
TTTT FFTT TTTT TTTT (2.1.i.2)

(ii) U is closed under finite intersections; if X ∈ U and Y ∈ U then X ∩ Y ∈ U. (2.1.ii.1)

((r<q)&(s<q))>((r&s)<q) ;
TTTT TTTT TTTT TTTT (2.1.ii.2)

(iii) I ∈ U but ∅ ∉ U. (2.1.iii.1)

(p<q)&~((p@p)<q) ; FTFF FTFF FTFF FTFF (2.1.iii.1)

Remark 2.1.0.2.1: We write Eq. 2.1.0.2.1 to imply 2.1.i.1 and 2.1.ii.1 and 2.1.iii.1. (2.1.0.3.1)

(((p=~(p@p))>(q>p))>(p>(p>p)))>(((((r<q)&~(~(p<s)<r))>(s<q))&(((r<q)&(s<q))>((r&s)<q)))&((p<q)&~((p@p)<q))) ;
FTFF FFFF FTFF FFFF (2.1.0.3.2)

An **ultrafilter** over I is a proper filter U over I such that: (2.1.0.4.0)

(iv) For each X ⊆ I, exactly one of the sets X, I \ X belongs to U. (2.1.iv.1)

~(p>#r)>((%q&(%p\%r))<q) ;
TFTF TNTN TFTF TNTN (2.1.iv.2)

Remark 2.1.0.4.0: We write Eq. 2.1.0.4.0 as 2.1.0.3.2 to imply 2.1.iv.1. (2.1.0.4.1)

((((p=~(p@p))>(q>p))>(p>(p>p)))>(((((r<q)&~(~(p<s)<r))>(s<q))&(((r<q)&(s<q))>((r&s)<q)))&((p<q)&~((p@p)<q))))>(~(p>#r)>((%q&(%p\%r))<q)) ;
TFTT TTTT TFTT TNTT (2.1.0.4.2)

Remark Def.2.1: The **proper filter** definition from six equations is *not* tautologous. The **ultrafilter** definition therefrom in two equations is *not* tautologous. This refutes the Keisler **ultraproduct** construction definition as "a uniform method of building models of first order theories which has applications in many areas of mathematics."

What follows is that claims to extend Keisler's construction also fail, as for example:

Malliaris, M.; Shelah, S. (2019). Keisler's order is not simple (and simple theories may not be either) arxiv.org/pdf/1906.10241.pdf

Abstract. Solving a decades-old problem we show that Keisler's 1967 order on theories has the maximum number of classes. The theories we build are simple unstable with no nontrivial forking, and reflect growth rates of sequences which may be thought of as densities of certain regular pairs, in the sense of Szemerédi's regularity lemma. The proof involves ideas from model theory, set theory, and finite combinatorics.

Text. Keisler's order is a longstanding classification problem in model theory, introduced in 1967 .. as a possible way of comparing the complexity of theories. … In the present paper we prove, in ZFC, that Keisler's order has the maximum number of classes (continuum many), by constructing a new family of simple unstable theories with no nontrivial forking which reflect growth rates of certain sequences of densities of finite graphs, and by developing new methods for building ultrafilters on Boolean algebras which carefully reflect these theories.