Refutation of first-order proofs without syntax

Abstract: Proof examples (3) in the introduction, modal combinatorial proofs (1), and rules in Gentzen's classical sequent calculus (3) are *not* tautologous. This refutes the conjecture and approach of first-order proofs without syntax, to form a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·; \ Not And; Proof examples (3) in the introduction, modal combinatorial proofs (1), and rules in Gentzen's classical sequent calculus (2) are *not* tautologous. This refutes the conjecture and approach of first-order proofs without syntax.

> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) N as non-contingency, Δ, ordinal 1; (%z<#z) C as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

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Abstract: Proofs are traditionally syntactic, inductively generated objects. This paper reformulates first-order logic (predicate calculus) with proofs which are graph theoretic rather than syntactic. It defines a *combinatorial proof* of a formula ϕ as a lax fibration over a graph associated with ϕ. The main theorem is soundness and completeness: a formula is a valid if and only if it has a combinatorial proof.

1 Introduction

Proofs are traditionally syntactic, inductively generated objects. For example, Fig. 1 shows a syntactic proof of

∃x(px ⇒ ∀y py). (1.1)

LET p, q, r, s: p, x, y, f (or a)

(p&%q)>(p&#r) ; TNT**F** TNTN TNT**F** TNTN (1.2)

The four combinatorial proofs of Fig. 2 are rendered in condensed form in Fig. 3.

(∀xpx) ⇒ ∀y (py∧pfy) (3.1.1)

(p&#q)>((p&#r)&(p&(s&#r))) ;
TTTC TTTC TTTC TTTT (3.1.2)

$\exists x(pa \vee py \Rightarrow px))$ (3.4.1)

((p&s)+(p&r))>(p&%q) ; `TTTT TCTT TCTT TCTT` (3.4.2)

9 Modal combinatorial proofs

A *modal* formula is generated from the *modal operators* □ (necessity) and ◊ (possibility) instead of quantifiers and has all predicate symbols nullary, *e.g.* ◊(p ⇒□p). Every modal formula abbreviates a standard first-order one .. : replace every □ by $\forall$x, ◊ by $\exists$x, and predicate symbol p by px. For example,

◊(p ⇒□p) abbreviates $\exists x(px \Rightarrow \forall x\, px)$, or $\exists x(px \Rightarrow \forall y\, py)$ in rectified form. (9.1)

%(p>#p) > (((p&%q)>(p&#q))+((p&%q)>(p&#r))) ;
`TNTN TNTN TNTN TNTN` (9.2)

11 Proof of the Completeness Theorem

In this section we prove the Completeness Theorem … . Our strategy will be to show that every syntactic proof of a formula ϕ in Gentzen's classical sequent calculus .. generates a combinatorial proof of ϕ, so completeness follows from that of Gentzen's system.

$$\frac{\Gamma}{\Gamma,\phi}\quad \text{W} \qquad (11.4.1)$$

LET p, q, r, s: ϕ, θ (or t in $\exists$), Γ, Δ (or x in $\forall$).

r>(r&p) ; T**F**TT T**F**TT T**F**TT T**F**TT (11.4.2)

$$\frac{\Gamma,\ \phi\{x7\rightarrow t\}}{\Gamma,\ \exists x\phi}\quad \exists \qquad (11.8.1)$$

((r&p)&(s>q))>(r&(%s&p)) ;
`TTTT TCTC TTTT TTTT` (11.8.2)

$$\frac{\Gamma,\phi}{\Gamma,\forall x\phi}\quad \forall\ (\text{x not free in } \Gamma) \qquad (11.9.1)$$

(r&p)>(r&#p) ; `TTTT TNTN TTTT TNTN` (11.9.2)

The rules X, C and W are called *exchange*, *contraction* and *weakening*. Each sequent above a rule is a *hypothesis* of the rule, and the sequent below a rule is the *conclusion* of the rule.

Proof examples (3) in the introduction, modal combinatorial proofs (1), and rules in Gentzen's classical sequent calculus (3) are *not* tautologous. This refutes the conjecture and approach of first-order proofs without syntax.