Refutation of formalization of axiom of choice and equivalent theorems using the Coq tool

© Copyright 2019 by Colin James III All rights reserved.

Abstract: We evaluate two definitions for maximum and minimal set membership, for the nesting of sets, and the equivalence relations of axiom of choice, Tukey's lemma, Hausdorff maximal principle, maximal principle, Zermelo's postulate, Zorn's lemma, well-ordering theorem. None is tautologous, refuting the claims. The authors conclude: "The whole process of formal proof demonstrates that the Coq-based machine proving of mathematics theorem is highly reliable and rigorous. The formal work of this paper is enough for most applications, especially in set theory, topology and algebra." We refute those assertions based on the non-bivalent performance of the Coq proof assistant. Therefore, these formalizations and methodology render a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated proof value, **F** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·; \ Not And;
> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) `T` as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥, zero;
(%z>#z) `N` as non-contingency, Δ, ordinal 1; (%z<#z) `C` as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: Sun, T.; Yu, W. (2019). Formalization of the axiom of choice and its equivalent theorems. arxiv.org/pdf/1906.03930.pdf {stycyj,wsyu}@bupt.edu.cn

Abstract. In this paper, we describe the formalization of the axiom of choice and several of its famous equivalent theorems in Morse-Kelley set theory. These theorems include Tukey's lemma, the Hausdorff maximal principle, the maximal principle, Zermelo's postulate, Zorn's lemma and the well-ordering theorem. We prove the above theorems by the axiom of choice in turn, and finally prove the axiom of choice by Zermelo's postulate and the well-ordering theorem, thus completing the cyclic proof of the equivalence between them. The proofs are checked formally using the Coq proof assistant in which Morse-Kelley set theory is formalized. The whole process of formal proof demonstrates that the Coq-based machine proving of mathematics theorem is highly reliable and rigorous. The formal work of this paper is enough for most applications, especially in set theory, topology and algebra.

The following definitions are very important, and they are used in Tukey's lemma, the Hausdorff maximal principle and so on.

Definition 3.2 (Maximal (Minimal) Member)**.** *F* is a maximal (minimal) member of *f* iff no member of *f* properly contains *F* (no member of *f* is properly contained in *F*). When *f* is equal to empty, *f* has no maximal (minimal) member. Thus we add the condition *f* ≠ Ø *;* when we formalize the maximal (minimal) member. The condition is very important in proving the existence of maximal elements in a set. At the same time, it eliminates many unnecessary discussions.

Definition MaxMember (F f: Class) : Prop : f ≠ Ø*;* -> F ∈ f ∧ (∀ E, E ∈ f -> ~ (F ⊊ E)).

(3.2.1.1)

LET p, q, r, s: f, E, F, s.

(p@(s@s))>((r<p)&((#q<p)>~(r<#q))) ;

T**F**T**F** T**F**T**F** T**F**T**F** T**F**T**F** (3.2.1.2)

Definition MinMember (F f: Class) : Prop : f ≠ Ø; -> F ∈ f ∧ (∀E, E ∈ f -> ∼ (E ⊊ F)).

(3.2.2.1)

(p@(s@s))>((r<p)&((#q<p)>~(#q<r))) ;

T**F**T**F** T**F**T**F** T**F**T**F** T**F**T**F** (3.2.2.2)

The following is the definition of nest. It will be used in the description of the Hausdorff maximal principle. The specific description and Coq formalization of it are as follows:

Definition 3.3 (Nest)**.** *n* is a nest if and only if, whenever *x* and *y* are members of *n*, then either $x \subset y$ or $y \subset x$.

Definition Nest n : Prop := ∀ x y, x ∈ n ∧ y∈ n -> x⊂ y ∨ y⊂ x. (3.3.1.1)

((#p<r)&(#q<r))>((#p<#q)&(#q<#p)) ;

TTTC TTTT TTTC TTTT (3.3.1.2)

Remark 3.3: Eqs. 3.2.1.2, 3.2.2.2, and 3.3.1.2 as rendered are *not* tautologous. This leads us to abandon mappings of subsequent equations with Coq-unique commands such as Ensemble, etc.

4. Formal proof of the equivalence

In this section, we present the formal proof of AC and its equivalent theorems. As shown in Figure 1, we start from AC to prove Tukey's lemma, the Hausdorff maximal principle, the maximal principle, Zermelo's postulate, Zorn's lemma and the well-ordering theorem in turn. We prove AC through Zermelo's postulate and the well-ordering theorem finally, thus completing the cyclic proof of the equivalence between AC and these theorems. Before each theorem is proved, we will give its formal description.

Figure 1: The relation of AC and its equivalent theorems

Axiom of Choice → Tukey's Lemma → Hausdorff Maximal Principle → Maximal Principle →

Zermelo's Postulate → Axiom of Choice

[or]

Zorn's Lemma → Well-ordering theorem → Axiom of Choice (4.0.1)

LET p axiom of choice; q Tukey's lemma; r Hausdorff maximal principle;
s maximal principle; t Zermelo's postulate; u Zorn's lemma; v well-ordering theorem.

((p>q)>(r>s))>(t>p))+((u>v)>p)) ;

TTTT TTTT TTTT TTTT}x1
FT**F**T TTTT **F**T**F**T **F**T**F**T}
TTTT TTTT TTTT TTTT(2)
TTTT TTTT TTTT TTTT}x2
FT**F**T TTTT **F**T**F**T **F**T**F**T} (4.0.2)

Remark 4.0: Eq. 4.0.2 as rendered is *not* tautologous, refuting the claimed relation of AC and its equivalent theorems using the Coq-assistant.