Discrete motives for moonshine

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Abstract

From the holographic perspective in quantum gravity, topological field theories like Chern-Simons are more than toy models for computation. An algebraic construction of the CFT associated to Witten's j-invariant for $2+1$ dimensional gravity aims to compute coefficients of modular forms from the combinatorics of quantum logic, dictated by axioms in higher dimensional categories, with heavy use of the golden ratio. This paper is self contained, including introductory material on lattices, and aims to show how the Monster group and its infinite module arise when the automorphisms of the Leech lattice are extended by special point sets in higher dimensions, notably the 72 dimensional lattice of Nebe.

Contents

1 Introduction

Given its definition in terms of Eisenstein series and their divisor functions, there is no doubt that the j-invariant $j(q)$ carries interesting number theoretic properties, as well as a module structure for the Monster group. Monstrous moonshine [1][2] employs a vertex operator algebra for bosonic strings, but a condensed matter approach is more appropriate for axiomatic quantum gravity, and stringy dimensions simply denote the number of strands for a ribbon diagram in categories for quantum computation. The 27 dimensions of bosonic M theory is the dimension of a state space for three qutrits, which we will describe. An axiomatic framework promises to shed light on the basic structure of modular forms, along with the operads underlying vertex operator algebras.

Borcherd's formula states that

$$\frac{j(q)-j(p)}{\frac{1}{q}-\frac{1}{p}} = \prod_{n,m\geq 1}^{\infty}(1-q^n p^m)^{c(nm)} = \prod_{N\geq 1}\prod_{d|N}(1-q^d p^{N/d})^{c(N)} \tag{1}$$

for d a divisor of N, where $c(i)$ is an integer coefficient of j. Recently in [3] it was used to study a duality between the inverse temperature in the usual argument q of j and a chemical potential associated to the number of copies k of j defining the CFT at $c = 24k$, where Witten's j-invariant [4] gives the Bekenstein-Hawking entropy for BTZ black holes at $c = 24$. The partition function is $\sum_{k\geq 0} p^{k+1} Z_k(q)$, where $Z_k(q) = \mathrm{Tr} q^{\Delta-k}$. The thermal AdS regime in this theory is characterised by Bose-Einstein condensate ground states, and a BEC screening of central charge. This additional parameter p has its own copy of $SL_2(\mathbb{Z})$. Here $c(n)$ is the number of states with $\Delta = n+1$ in Witten's CFT.

The q and p variables in [3] are interchanged by a $\mathbb{Z}_2$ symmetry, implying a correspondence between temperature and minimal AdS_3 masses. We expect such behaviour in the Fourier supersymmetry [5] between massive neutrinos and CMB photons, under which the present day CMB temperature corresponds precisely to a neutrino mass. In the ribbon spectrum, all Standard Model particles are based on the neutrino diagram.

Borcherd's formula is closely related [6] to $\sum_{n\geq 0} j_n(q) p^n$, where p is our second variable and $j_n(q)$ is derived from $j(q)$ by action of the Hecke operator $nT_{n,0}$ of weight zero, with $j_1(q)$ our usual j function. For $f(q) = \sum a(n) q^n$, this is the operator

$$T_{n,0} f(q) \equiv \sum_{i\geq 0} (a(ni) + a(i/n)/n) q^n, \tag{2}$$

assuming that $a(i/n)$ is only non zero for integral arguments.

For us, dualities and trialities are always information theoretic [7][8], and Bose-Einstein condensation is fundamental to the localisation of mass in the cosmological neutrino vacuum, for which the quantum neutrino IR scale and its dual Planck scale underpin the Higgs mechanism [9]. The supersymmetry between Standard Model fermions and bosons introduces the 24 dimensions of the Leech lattice as hidden structure for the Z boson, which is related to the non local states of the neutrino. As is well known, square roots of rest masses come in eigenvalue triplets with simple parameters, and we hope to complete the rest mass derivations with the scale ratios, starting with a $\sqrt{Z/\nu_R}$ ratio close to the number of short vectors for a certain lattice in high dimensions.

There are no classical string manifolds and no supersymmetric partners. In fact, the quantum topos perspective aims to reformulate $\mathbb{R}$ and $\mathbb{C}$ using higher dimensional categories, matching the combinatorics of categorical polytopes to generalised discrete root systems using canonical rings to define CFTs.

We leave further discussion of the Monster CFTs to the last section, starting with the combinatorial structure of lattices and the j-invariant, motivated by motivic quantum gravity. The next few sections are quite elementary, for readers unfamiliar with modular mathematics, but we begin with some pertinent remarks on set theory, which are crucial to the category theoretic philosophy and its implications for the underlying motivic axioms.

2 Sets in quantum gravity

In order to build a CFT for a holographic theory, we want to throw a great deal of higher dimensional information into two dimensions. For instance, the roots of any Lie algebra are projected onto the so called magic star [10], which extends in exceptional periodicity [11] beyond the exceptional Lie algebras using a broken Jacobi rule for T-algebras. Broken rules for algebras make perfect sense in operads, which come with an infinite tower of rules for operad composition. The L_∞ operad [12] replaces Lie algebras when we are focused on homotopy, which secretly we are.

We are also interested in the quasi-lattices generated by projections to lower dimensions, particularly the two dimensional Penrose quasi-lattice [13], which employs the golden ratio $\phi = (1+\sqrt{5})/2$ to embed $\mathbb{Z}^8/2$ in the plane, including of course the roots of $\mathbf{e_8}$. All these point sets may be constructed from a few simple polytopes, mostly permutohedra and cubes, which carry canonical algebraic labels as categorical axioms.

The category theory is also essential for another reason: physical measurements are statements in quantum logic, where a dimension of a Hilbert space replaces the classical cardinality of a set. Thus quantum mechanics forces us immediately into an infinite dimensional setting. To make a measurement, we must also account for the classical data, whose logic is governed by the category of sets. It is therefore perfectly natural to map the subset lattice for an n point set in the category of sets onto a cube in n dimensions. Such cubes are fundamental as targets of the power set functor on the category of sets, dictating Boolean logic in the topos [14].

For our generalised Lie algebras, the spinor dimensions in the $\mathbf{e_8}$ chain of exceptional periodicity [11] go up by factors of 16, as in $8, 128, 2048, 32768$ for the first four levels. Each 2^{4n-1} counts the number of vertices on a cube in dimension $4n-1$. In particular, 2^3 gives the charges of leptons and quarks, just as it defines a basis for $\mathbb{O}$, and 2^7 traditionally carries magnetic data. If we take a basis for a Hilbert space, the cube contains both basis vectors and all other subsets of the basis set. For example, we label the 8 vertices of the three dimensional cube

$$1, e_1, e_2, e_3, e_1e_2, e_2e_3, e_3e_1, e_1e_2e_3. \tag{3}$$

Here 1 denotes the empty set. The XOR product on subsets (either A or B but not both) is addition in the Boolean ring with intersection as product [15]. This recovers the structure of the Fano plane in ${\mathbb{F}_2}^3$, and hence the units in $\mathbb{O}$ [16]. Intersection is defined using the arrows and faces of the cube. Our subsets are also denoted by the $\mathbb{F}_2$ sign strings, so that e_1 is $+--$.

Given the recovery of a dense set in $\mathbb{C}$ using only the ring $\mathbb{Q}(\sqrt{\phi+2})$ [13], we dispose of the usual continua in favour of more topos friendly constructions. The reals are equal in cardinality to the power set of $\mathbb{N}$, which is just a second application of the power set monad to our initial category, picking out dimensions $n = 2^k$. In exceptional periodicity [11] this picks out the spatial dimensions 8 and 32, where four copies of $\mathbb{Z}^8/2$ are needed for $SL_2(\mathbb{C})$, the cover of the Lorentz group. The infinite cube defines reals, within the surreals, that are not finite dyadic: an infinite string of minus signs is the infinitesimal ϵ and the infinite plus string is the surreal ω. Thus the surreals, or something like them, assign a natural normalisation of 2^{-n} to n qubits.

When each e_i coordinate of (3) is extended to a discrete line $e_i, e_i^2, \cdots$, as if the line is a path space, then coordinates represent the prime factors of $N = \prod_{i=1}^r p_i^{k_i}$ for $N \in \mathbb{N}$, in a general rectangular array of points. All divisors d of N sit on the points below N, which is the target of the rectangular block. For example, $N = 30 = 2\cdot3\cdot5$ is modeled on the basic 3-cube, and each square free N gives a parity cube in dimension r. In this picture, $\mathbb{Z}$ is an infinite dimensional cubic cone, with a discrete axis for each prime. This is natural, because the Cartesian product $N \times M$ has either NM points or $N + M$ dimensions as a vector space. Taking all subsets of a basis enumerates all possible sets of linearly independent vectors within the basis.

Sign strings of length n arise as signature classes for permutations in S_{n+1}. For example, (2314) in S_4 belongs to the class $+-+$, with a plus denoting an increase in numerals as we read the permutation left to right. Eight vertices on the 3-cube are therefore derived from the 24 vertices of the S_4 permutohedron, a polytope in dimension 3. A group algebra $\mathbb{F}S_{n+1}$ descends to the Solomon Hopf algebra on the vertices of the cube, starting with the elements $\sum_i \pi_i$ for $\pi_i \in S_{n+1}$ ranging over the signature class. The vertices of the permutohedron, which also tiles three dimensional Euclidean space, are denoted by the divisors of the number $N = {p_1}^n {p_2}^{n-1} {p_3}^{n-2} \cdots p_n$. For S_4, we get the 24 points

$$\begin{aligned} 1, p_1, p_2, p_3, {p_1}^2, {p_2}^2, {p_1}^3, \\ p_2p_3, p_1p_3, p_1p_2, p_3{p_2}^2, p_3{p_1}^2, p_2{p_1}^2, p_1{p_2}^2, p_2{p_3}^2, \\ p_2{p_3}^3, {p_2}^2{p_3}^3, {p_2}^2{p_3}^2, p_1p_2p_3, \\ p_1{p_2}^2p_3, p_1{p_2}^2{p_3}^2, p_1{p_2}^2{p_3}^3, p_1p_2{p_3}^2, p_1p_2{p_3}^3, \end{aligned} \tag{4}$$

viewed as permutations of $(-3, -1, 1, 3)/2$ in $\mathbb{Z}^4/2$, or more often, $(1, 2, 3, 4)$. This permutohedron is mapped to the $(1, 0, 0, 1)$ coordinates of the 24-cell as follows. A $(1,2,3,4)$ vector sends the 3 and 4 to 1, and the 1 and 2 to 0. Since each parity square face on the S_4 polytope has 3 and 4 in the same positions, the square adds the signs to the resulting vector. Extending the 24-cell coordinates into 8 dimensions using four extra zeroes, we get as usual the 112 bosonic roots of $\mathbf{e_8}$. Opposite pairs of squares on S_4 give three separate copies of the 3-cube basis for $\mathbb{O}$, so that 9 copies of S_4 provide sets of 24 roots on the 7 internal points of the magic star in the plane. One further copy of S_4 catalogs (i) 6 points for the $\mathbf{a_2}$ hexagon and (ii) three $J_3(\mathbb{O})$ diagonal elements on each tip of the star.

Replacing sets and permutations by vector spaces and endomorphisms, for quantum logic, we naturally consider the vector space analog of transpositions, namely reflections. This is why generalised root lattices appear naturally in quantum gravity.

3 Icosians and the permutohedron

Let $\phi = (1+\sqrt{5})/2$ be the golden ratio. The Leech lattice is easily defined in terms of the 120 norm 1 icosians (38) in $\mathbb{H}$. As with the $\mathbb{O}^3$ Leech lattice, the icosian lattice is a subset of vectors $(x,y,z) \in \mathbb{H}^3$ with x, y and z all icosians. First consider the 24 icosians

$$\pm 1, \pm i, \pm j, \pm k, \frac{1}{2}(\pm 1 \pm i \pm j \pm k). \tag{5}$$

Since a basis $\pm\{1,i,j,k\}$ forms a parity square in the plane, as described in the last section, we would like to group these 24 icosians into three pairs of squares, such that the ones with three minus signs belong to one square. These sets are not orthogonal (in sets of three) in $\mathbb{H}$, but they are in $\mathbb{H}^3$ if we spread the squares out into different copies of $\mathbb{H}$. Then we project from 12 dimensions down to 3, obtaining three pairs of square faces on the permutohedron S_4. For example, the square

$$-1+i+j+k, \;\; 1+i-j+k, \;\; 1-i+j+k, \;\; 1+i+j-k \tag{6}$$

starts with $++$ on i, j, and flips i or j on its diagonal.

The other 96 icosians of interest are of the form

$$\frac{1}{2}(\pm i \pm \phi j \pm \phi^{-1} k) \tag{7}$$

up to signs and even permutations on $\{1,i,j,k\}$. That is, 8 copies of A_4 in $\pm$ pairs.

In total, there are 120 such icosians of norm 1, suggesting 5 copies of the non standard permutohedron. There is a natural way to take 5 copies of a permutohedron in dimension 3 to build a 120 vertex polytope known as the permutoassociahedron [17], obtained by replacing each vertex on S_4 with a pentagon. It is crucial to the axioms for ribbon categories.

4 The Fibonacci reflection and braids

The simplest expression for the j-invariant in (15) is invariant under the S_3 permutations of three roots for a cubic, where group multiplication is function substitution under the correspondence

$$z = (1), \;\; 1 - \frac{1}{z} = (312), \;\; \frac{1}{1-z} = (231), \tag{8}$$

$$\frac{1}{z} = (31), \;\; \frac{z}{z-1} = (32), \;\; 1 - z = (21).$$

This is almost, but not quite, represented by matrices in the modular group $\Gamma = PSL_2(\mathbb{Z})$. Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tag{9}$$

be the two generators of Γ and let

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10}$$

be a reflection across the axis in the plane. Then the correct representation [18] is

$$(1) = I, \;\; (312) = TS, \;\; (231) = ST^{-1}, \tag{11}$$

$$(31) = -SZ, \;\; (32) = -ST^{-1}SZ, \;\; (21) = -TZ.$$

We have to multiply by Z on the right to obtain the usual $PSL_2(\mathbb{F}_2)$, for which $T^2 = I$. Now observe that the powers $(TSZ)^n$ generate the Fibonacci numbers F_k [18]. At mod m, the set of F_k is a cycle of length $L(m)$, depending on primes associated to k. In particular, we have F_4 mod $3 \equiv 0$, so that $T^8 = I$ and $(TSZ)^4 = I$, giving a representation of S_4. For $PSL_2(\mathbb{F}_7)$ we need F_k mod 13. Up to mod 12, all cycles fit into a cycle of length 240.

Recall that the limit of F_{k+1}/F_k is the golden ratio $\phi = (1 + \sqrt{5})/2$. In order to understand the relationships between different structures on the Leech lattice, we need to look at the rotation between normed division algebra braids and Fibonacci anyon representations [19]. The cyclic braid group B_3^c on three strands [19][20] is given by quaternion units i, j, k in

$$\sigma_1 = \frac{1}{\sqrt{2}}(1 + i), \;\; \sigma_2 = \frac{1}{\sqrt{2}}(1 + j), \;\; \sigma_3 = \frac{1}{\sqrt{2}}(1 + k), \tag{12}$$

such that $\sigma_i^8 = 1$. The complex B_2 generator σ_1 is used to describe ribbon twists in an extension of B_3^c to ribbon diagrams for particle states. Similarly, B_7^c has a representation [20]

$$\sigma_1 = \frac{1}{\sqrt{2}}(1 + e_2e_1), \;\; \sigma_2 = \frac{1}{\sqrt{2}}(1 + e_3e_2), \;\; \sigma_3 = \frac{1}{\sqrt{2}}(1 + e_4e_3), \tag{13}$$

$$\sigma_4 = \frac{1}{\sqrt{2}}(1 + e_5e_4), \;\; \sigma_5 = \frac{1}{\sqrt{2}}(1 + e_6e_5), \;\; \sigma_6 = \frac{1}{\sqrt{2}}(1 + e_7e_6),$$

$$\sigma_7 = \frac{1}{\sqrt{2}}(1 + e_1 e_7),$$

where the e_i Clifford algebra elements all satisfy $e_i^2 = -1$ and $e_i e_j = -e_j e_i$. This B_7^c will appear in associative algebras based on $\mathbb{O}$.

The B_3 representation based on j and k, viewing $\mathbb{H}$ as 2×2 matrices, is rotated to a matrix Fibonacci representation in $SU(2)$ [19] using

$$M = e^{7\pi j/10}, \quad P = j\phi + k\sqrt{\phi}, \quad N = PMP^{-1}, \tag{14}$$

satisfying the braid relation $MNM = NMN$. This rotation is 9°, where $(\phi\sqrt{\phi+2})^{-1} = \tan 18^\circ$ and $\phi = 2\cos(2\pi/10)$. The basic arithmetic phase $\pi/12$ enters in $\pi/10 - \pi/12 = 36^\circ$, half the pentagon angle.

5 The j-invariant and golden ratio

Before discussing the Leech lattice, we introduce the j-invariant for the modular group Γ. A divisor function of k-th powers is denoted $\sigma_k(n)$. Let $z = a + ib$ and

$$J(z) = 1728 \cdot \frac{4}{27} \cdot \frac{(z^2 - z + 1)^3}{z^2(z-1)^2}, \tag{15}$$

so that $J(i) = 1728$. It's famous Fourier expansion is

$$j(q) = J(q) - 744 = q^{-1} + 196884q + 21493760q^2 + \cdots \tag{16}$$

Let's imagine we are interested in real values of the j-invariant. Using the numerator and denominator of (15), we define a 2×2 matrix using the coeffients of

$$J(z) = \frac{A + iB}{C + iD}. \tag{17}$$

Such a splitting of terms covers two cases of interest: (i) A, B, C, D all real, and (ii) A, C rational and B, D pure imaginary irrational. The reality condition is

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = 0, \tag{18}$$

giving a degree 9 or 10 polynomial P in a and b. At $a = 1$ we obtain

$$P(b) = b^3(b^2 - 1)(b^2 - ib + 1)(b^2 + ib + 1), \tag{19}$$

which defines the critical values

$$0, \pm 1, \pm\phi, \pm\phi^{-1}. \tag{20}$$

For small values of b, we pick up the standard critical values

$$0, \pm 1, \infty, \frac{1}{2}, 2, e^{\pi i/3}, e^{-\pi i/3} \tag{21}$$

of the ribbon graph, as roots of

$$P(a) = a(a-1)(a+1)(a-2)(2a-1)(a^2-a+1)^2. \tag{22}$$

Note the invariance under $a \mapsto 1-a$, and

$$J(\pm\phi) = J(\pm\phi^{-1}) = 2048 = 2^{11}. \tag{23}$$

This golden ratio value appears in the combinatorics of the first shell of the Leech lattice Λ, where $\sigma_{11}(2) = 2049$ appears in the second term of the lattice form

$$f_\Lambda = \sum_{i=0}^{\infty} \frac{65520}{691}(\sigma_{11}(i) - \tau(i)) = 1 + 196560q^2 + 16773120q^3 + \cdots \tag{24}$$

which includes Ramanujan's τ coefficients for the modular discriminant

$$\Delta(q) = \sum_{n=1}^{\infty} \tau(n)q^n = q - 24q^2 + 252q^3 - \cdots \tag{25}$$

The nicest Leech lattice integers appear with the Hecke operator T_2, which acts on the dimension 2 space of weight 12 forms for Γ, giving

$$T_2(f_\Lambda) = 2049 + 196560q + \cdots, \quad T_2(\Delta) = 0 - 24q + \cdots \tag{26}$$

For quadratic fields over $\mathbb{Q}$, the only good integral values of $j(z)$ are the twelve critical values listed in (20) and (21), as follows. Let $z = a\sqrt{n} + b$ for $a, b \in \mathbb{Q}$. When the numerator and denominator of j are rational, we look for a numerator that is a multiple of the denominator, in the form

$$(64x^2 - 32x + 4)(x + s) \tag{27}$$

with $x = a^2n$. The constant term forces $s = 27/4$ and we find the solution $x = 5/4$. Defining $y = 4a^2n$ and seeing that j is proportional to

$$\frac{(y+3)^3}{(y-1)^2}, \tag{28}$$

it must be that the prime factors of $y - 1$ are obtained in the factors of the numerator, restricting us to $p = 2$. This forces the solution $a = \pm 1/2$, $n = 5$, giving $z = \phi$ or ϕ^{-1}. Consider now

$$f(a\sqrt{n} + b) \equiv a^2n + b^2 + (2ab - a)\sqrt{n} - b \tag{29}$$

in the numerator. There must exist $c \in \mathbb{Q}$ such that $f^3 + 3cf + c = 0$, but then f^2 is also rational, which is only the case for n a square. Siegel's theorem [21] states that, besides $\mathbb{Q}$, the only (totally real) algebraic number field for which every ordinal is a sum of at most three squares is $\mathbb{Q}(\sqrt{5})$, which contains the golden ratio integers.

Let $C(n)$ be the Fourier coefficients of $j(q)+24$, using (16). A recursion formula [22] for $C(n)$ is

$$C(n) = -\sum_{i=-1}^{n-1} C(i)\tau(n+1-i) + \frac{65520}{691}(\sigma_{11}(n+1) - \tau(n+1)). \tag{30}$$

In particular,

$$C(1) = 196884 = 24^2 - 252 + 196560 \tag{31}$$

reminds us of the Monster. The $C(n)$ coefficients exhibit Ramanujan type congruences [23] such as

$$C(5^i k) \equiv 0 \mod 5^{i+1}, \quad C(7^i k) \equiv 0 \mod 7^i, \quad C(11k) \equiv 0 \mod 11, \tag{32}$$

where i is any positive ordinal. Here $5,7,11$ are equal to $n+3$ for $n=2,4,8$, the dimensions of the division algebras. When talking about $j(q)$, we will denote the coefficients by little $c(n)$.

6 Cubes and the Leech lattice

We look at the Leech lattice Λ for the octonions $\mathbb{O}$, for the icosians in $\mathbb{H}$, and for $\mathbb{C}$. Later we will also look at a $\mathbb{Z}[\alpha]$ structure where, once and for all, we fix $\alpha = (1+\sqrt{-7})/2$. Table 1 gives the octonion multiplication table following [24], which is close to subset notation.

Table 1: multiplication of units in $\mathbb{O}$

	i	j	k	il	jl	kl	l
i	-1	k	$-j$	$-l$	kl	$-jl$	il
j		-1	i	$-kl$	$-l$	il	jl
k			-1	jl	$-il$	$-l$	kl
il				-1	k	$-j$	$-i$
jl					-1	i	$-j$
kl						-1	$-k$
l							-1

The Leech lattice is described in terms of integral octonions in $\mathbb{O}^3$ by Wilson [25][26]. Start with the 8 dimensional root lattice L_8 of $\mathbf{e_8}$, generated by a set of 240 unit octonions. These are the 112 octonions of the form $\pm e_i \pm e_j$ for any distinct units e_i and e_j of $\mathbb{O}$, and the 128 octonions of the form $(\pm 1 \pm i \pm j \pm \cdots \pm l)/2$ with an odd number of minus signs. We write $L = L_8$. We could start instead with a right lattice R, or the lattice $2B = LR$, and this will be discussed in section 8. The Eisenstein form E_4 is the norm function for L_8, counting the vectors of length $2n$. Let $s = (-1+i+j+\cdots+kl+l)/2$. Then the Leech lattice Λ [25] is the set of all triplets $(u,v,w) \in \mathbb{O}^3$ such that

1. u, v, $w \in L$

2. $u+v,\ v+w,\ w+u \in L\overline{s}$

3. $u+v+w \in Ls$.

Note that $L\overline{s} \cap Ls = 2L$. For future reference, given a root X in L, Λ includes the norm 4 vectors

$$(X\overline{s}, X\overline{s}, 0), \quad (2X, 0, 0), \quad (Xs, X, X). \tag{33}$$

Consider the Leech vector

$$(1+i)(i+jl)s = -1-i-j-k-il-jl+kl-l \tag{34}$$

of the form $2X$, where $X \in L$. If we multiply X on the right by any of the 8 units we obtain other spinors in L, and hence 24 vectors of shape $(2X, 0, 0)$ in Λ. The number of short vectors in Λ is $196560 = 24 \cdot 8190$, where $819 = 3 \cdot 273$. The number

$$273 = 1 + 16 + 16^2 \tag{35}$$

accounts [25] for the sign choices for the charts of a projective plane $\mathbb{OP}^2$.

When Λ is embedded in the exceptional Jordan algebra $J_3(\mathbb{O})$ [27], triality acts on the off diagonal elements of

$$\begin{pmatrix} a & X & \overline{Y} \\ \overline{X} & b & Z \\ Y & \overline{Z} & c \end{pmatrix}, \tag{36}$$

for a, b, c real. Triality is given by a triple of maps: left, right and two sided multiplication (L, R, B) by elements of $\mathbb{O}$. We look further at triality in section 8.

Inside the 16 dimensional $\mathbb{O}^2$ lies an S^{15}, which can be associated to a discrete Hopf fibration $S^{15} \to S^8$ [28][31]. The number of short vectors also satisfies

$$196560 = 196608 - 48, \tag{37}$$

where $196608/2 = 3 \cdot 2^{15} = 24 \cdot 2^{12}$. That is, identifying opposite points in the sphere of short vectors in Λ takes us to the projective $\mathbb{OP}^2$, which has three charts based on $\mathbb{O}^2$. Each $\mathbb{O}^2$ contains a cube with 2^{15} vertices, and we subtract 8 basis points for each copy of $\mathbb{O}$ so that the 2-forms are minimal [28].

Compare this to the standard 24 vectors in the real Λ, which are of the form $(3, 1, 1, \cdots, 1)$, considered to be in $\mathbb{C}^{12}$, so that there are only 2^{12} sign choices. To remove 24 vectors, fix one positive sign on a 3 and make all remaining signs negative.

Instead of the usual $\mathbb{OP}^2$ decomposition $\mathbb{O}^2 + \mathbb{O} + 1$, we may wish to add a *regulator* triangle of three points near infinity on the affine plane. This is analogous to adding two points on a line in a triangle model for $\mathbb{RP}^1$, turning the triangle into the famous pentagon [29]. Adding three points to a discrete cohomological $\mathbb{RP}^2$ gets us the 14 vertex associahedron.

In the complex Λ [30] over $\mathbb{Z}[\omega]$, for ω the primitive cubed root, it is useful to use the vector $(\theta,\theta,\theta,\theta,\theta,\theta,0,0,0,0,0,0)$, where $\theta=\omega-\overline{\omega}$. Its integral norm is 18, which is adjusted down to 4 by a factor of $2/9$, a parameter that will often crop up, most notably in the $J_3(\mathbb{C})$ Koide mass matrices for leptons.

As noted above, the quaternion Leech lattice [26][32] uses the 120 norm 1 icosians of the form

$$\frac{1}{2}(\pm 1 \pm i \pm j \pm k), \quad \frac{1}{2}(\pm i \pm \phi^{-1} j \pm \phi k), \quad \pm 1, \quad \pm i, \quad \pm j, \quad \pm k, \tag{38}$$

representing $SL_2(5)$. In analogy to the $\mathbb{O}^3$ construction, consider vectors (x,y,z) in $\mathbb{H}^3$. Let $h=(-\sqrt{5}+i+j+k)/2$ and $\overline{h}$ its quaternion conjugate. The number h defines a left ideal in the icosians, as do the four other numbers given by an odd number of minus signs on (i,j,k) in h, which includes $\overline{h}$. These five ideals define the 600 norm 2 icosians. Let L_h and $L_{\overline{h}}$ denote the two ideals, given by left multiplication in the full icosian ring. Then Λ is the lattice defined by

1. $x,\ y,\ z \in L_h$
2. $x+y+z \in L_{\overline{h}}$.

Conway [33] looks at Lorentzian lattices in $\mathbb{R}^{d,1}$ for $d \equiv 1 \bmod 8$. In particular, at $d=25$ there exists an infinite group of automorphisms for the Leech lattice, including the translations, generated by the reflections of fundamental roots, where a root $u \in \Lambda$ satisfies $u \cdot u = 2$ and $u \cdot w = -1$ for the norm zero 26-vector

$$w = (0,1,2,\cdots,23,24,70). \tag{39}$$

In today's Atlas notation, the finite group $\mathrm{Aut}(\Lambda)$ is called $2 \cdot \mathrm{Co}_1$, where the simple Conway group Co_1 has order $25 \cdot 27 \cdot 196560 \cdot 2^8 \cdot |M_{24}|$, for M_{24} the Mathieu group.

Recall the braid group rotation (14). We want to use it to understand how the $\mathbb{O}^3$ structure on Λ relates to the $\mathbb{H}^3(\phi)$ icosian structure. Similarly, in 12 dimensions, we will relate the $\mathbb{Z}[\omega]$ structure to the $\mathbb{Z}[\alpha]$ one, but this requires a peek at the 72 dimensional lattice.

With the golden ratio and octonions appearing everywhere, we realise that 5 point bases are just as important as 3 point ones. After all, $SU(3)$ for color uses a 3-space and a 5-space. The permutohedron S_6 has 720 points in dimension 5. Taking the Cartesian product of a 5-space with $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$, we get the triplet of dimensions

$$5+2=7, \quad 5+4=9, \quad 5+8=13, \tag{40}$$

so that a tensor product of quantum components has dimension $7 \cdot 9 \cdot 13 = 819 = 3 \cdot 273$. This 819 counts the so called integral Jordan roots [34], which define a simplex for $\mathbb{OP}^2$ [35] which is distinct from the optimal simplices based on mutually unbiased bases. The idea is to think of the crux pitch in the Monster in terms of such canonical discrete spheres, leaving most of the group elements to the simple MUBs.

7 Constructing a Jordan algebra

For $\mathbb{RP}^n$ the important polytopes are the associahedra [29][36], which are naturally embedded in an n-dit cube of dimension n inside an $(n+1)$-dit space [37] as follows. Look at the diagonal simplex defined by paths of length n on a cubic lattice. Parking function words, which are noncommutative paths such that the order of the i-th letter is greater than or equal to i, fit onto a subset of vertices in the triangular simplex, which defines commutative monomials in path letters.

In particular, the pentagon sits inside the 10 point tetractys simplex for 27 paths on a 64 point 3-cube, while the associator edge carries 3 paths on the diagonal of a 9 point square. The target vertex on an associahedron carries a copy of S_n, with S_4 appearing on the 14 vertex polytope.

Consider the 27 length 3 paths in the letters X, Y and Z. Our trit letters might represent powers of the three matrices

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad W = \omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{41}$$

in a product $U^X V^Y W^Z$, where ω is the primitive cubed root of unity. The spatial matrix V generates C_3, while the momentum operator is obtained as its quantum Fourier transform. These 27 matrices define a discrete phase space, and are used to define the exceptional Jordan algebra $J_3(\mathbb{O})$ using $\mathbb{F}_3 \times {\mathbb{F}_3}^3$, as shown in [38].

These four trits arise in the following simplex. To properly separate the 27 components of the Jordan algebra, the tetractys needs to be replaced by a simplex carrying 81 paths of length 4. These paths live in a cube with $5^3 = 125$ vertices (which counts parking functions). A parity 3-cube for a basis of $\mathbb{O}$ appears when one selects one out of four letters. For instance, choosing $ZXXX$ out of four possible permutations marks the first letter for deletion, leaving the word XXX, so that a parity cube is now labelled

$$ZXXX, \{XXXZ, XXZX, XZXX\}, \{ZXZZ, ZZXZ, ZZZX\}, XZZZ. \tag{42}$$

On this 81 path simplex, the unused corners $XXXX$, $YYYY$ and $ZZZZ$ are free to define the diagonal of a 3×3 matrix, so that the boundary of the simplex (without edge centre points) gives the 27 dimensions of $J_3(\mathbb{O})$.

Observe that the central 54 paths, which are not included in our 27, reduce under the letter deletion operation either to existing paths on the parity cubes or to the six XYZ paths of S_3. In this way the spinor splitting $27 = 16 + 11$, which ignores the shadow 54, reduces on the tetractys tile to $12 + 9$, so that the tetractys centre S_3 is sourced from the shadow paths. Selecting 6 out of 27 tetractys paths is one way to obtain that factor of $2/9$, which occurs as a lepton Koide phase for rest mass triplets [9][5] in $\mathbb{C}S_3$. Below we will look at a 72 dimensional lattice, which uses three copies of the Leech lattice. If the leptons only see one copy, then this shadow 2/9 is related to the 2/9 that appears in the normalisation of the complex lattice, as noted in section 6.

An example of a 3-cube inside S_4 on the paths $ZZZY$ and $YYYZ$ is

$$YYY: 1324 \quad YYZ: 2314, 3142, 1423 \qquad (43)$$
$$ZZY: 2413, 4132, 3241 \quad ZZZ: 4231.$$

Thus 9 copies of S_4 fill 216 roots on the magic star for $\mathbf{e_8}$, leaving 18 diagonal entries plus 6 points for $\mathbf{a_2}$. On this tenth S_4, permutations are spread around the triangles of the star. For example, relative to an outer $\mathbf{a_2}$ hexagon that fixes a 1 in the first coordinate, one vertex on the star carries a 432 subcycle, for the other three possible positions of 1, giving the $J_3(\mathbb{O})$ diagonal $(4132, 4312, 4321)$. A triple of Jordan algebras is then the 1-circulant set $\{432, 243, 324\}$, which is the usual basis for Koide mass matrices.

In the magic $\mathbf{a_2}$ plane [10][11], a 27 dimensional $J_3(\mathbb{O})$ is assigned to each point on the star, as a piece of the 240 roots of $\mathbf{e_8}$. This plane is tiled by tetractys simplices, each carrying three pentagons, and a discrete blowup in the plane replaces a point on the star with a simplex. In the 64 vertex cube, which has a total of 1680 paths from source to target, there are actually two diagonals that hold a tetractys, their triangle boundaries pointing in opposite directions, so that the projection of these two simplices along the diagonal gives the magic star. Combining associahedra and permutohedra, we obtain the 120 vertex permutoassociahedron [17] in dimension 3, counting half the roots in $\mathbf{e_8}$. In 4 dimensions, a permutoassociahedron has $1680 = 5! \cdot 14$ vertices.

8 Triality and integral forms

In section 6 we defined the Leech lattice Λ using one of the lattices L, R and B. To understand the distinction between these lattices, we require the so called *integral octonions* $I = I(\mathbb{O})$ [39]. Let

$$q = a_0 + a_1 i + a_2 j + a_3 k + a_4 il + a_5 jl + a_6 kl + a_7 l \qquad (44)$$

be in $\mathbb{O}$, with norm $N(a) = \overline{a}a$. A set closed under addition and multiplication, with a 1, is *integral* if (i) $2a_0$ and $N(a)$ are in $\mathbb{Z}$, (ii) it is not contained in a larger such set. Thus the Gaussian $\mathbb{Z}[i]$ are integral in $\mathbb{C}$, and for $\mathbb{H}$ it is the 24 units of (5). In $\mathbb{O}$, an element $e = q_1 + q_2 l$ is defined in terms of two quaternions q_1 and q_2 using the unit l of Table 1. Now let $t = (i + j + k + l)/2$. Coxeter [39] then defines I in terms of the 8 elements

$$1, i, j, k, t, it, jt, kt, \qquad (45)$$

which close under multiplication. The 240 units include the 16 basis units, with a sign, 112 numbers of the form $(\pm 1 \pm j \pm l \pm jl)/2$ and 112 of the form $(\pm j \pm k \pm jl \pm kl)/2$ or $(\pm i \pm j \pm k \pm l)/2$.

Given I, the lattices L, R and $2B$ are defined by [26]

$$L = (1 + l)I, \quad R = I(1 + l), \quad 2B = (1 + l)I(1 + l). \qquad (46)$$

Here we see clearly the actions on each lattice, which may be embedded in the X, Y and Z components of $J_3(\mathbb{O})$. These are related to Peirce decompositions [40] for a noncommutative ring, splitting idempotents. The choice of one special octonion (in this case l) is used [9][41][42] to separate leptons from quarks in the $\mathbb{C} \otimes \mathbb{O}$ ideal algebra for Standard Model particles.

In section 6 we saw the number 819 as a factor of 196560. It appears now in the integral form of $J_3(\mathbb{O})$. Observe that the primes $7 = 4 + 2 + 1$ and $13 = 9 + 3 + 1$ count the points in projective planes for $\mathbb{F}_2$ and $\mathbb{F}_3$. The so called monomial subgroup G [26] of order $3 \cdot 2^{12}$ is generated by the maps

$$\begin{pmatrix} x & A & \overline{B} \\ \overline{A} & y & C \\ B & \overline{C} & z \end{pmatrix} \mapsto \begin{pmatrix} x & eA & \overline{Be} \\ \overline{eA} & y & eCe \\ Be & \overline{eCe} & z \end{pmatrix}, \tag{47}$$

and the permutations $(x, y, z; A, B, C) \mapsto (y, z, x; B, C, A)$ and $(x, z, y; \overline{A}, \overline{C}, \overline{B})$, for any unit e. Now let $s = (1 + i + j + \cdots + l)/2$ be the vector above. The group G acts on the elements

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \frac{1}{4}\begin{pmatrix} 2 & 1 & \overline{s} \\ 1 & 1 & \overline{s} \\ s & s & 1 \end{pmatrix}, \tag{48}$$

to give $819 = 768 + 48 + 3$ elements. The twisted finite simple group ${}^3D_4(2)$ is generated by Jordan reflections in these 819 roots, and has order $819 \cdot 63 \cdot 2^{12}$. The full automorphism group is the semidirect product ${}^3D_4(2) : C_3$ [26].

In constructing $F_4(2)$ and other finite groups, we often encounter the two dimensional discrete Fourier transform

$$F_2 \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{49}$$

The matrix $F_2 \otimes I_2$ maps short roots to long roots. The D_4 triality automorphism T is essentially given by $F_2 \otimes F_2$ in the form

$$T \equiv -\frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{50}$$

We see that $T^3 = 1$, as in the lattice triality $L \to R \to B$. The columns of T are the eigenvectors of the γ_5 matrix in the Dirac representation, making T one of the five mutually unbiased bases in dimension 4. Two copies of T are used in conjunction with ϕ to project the L_8 lattice down to a four dimensional quasi-lattice [43], in the $\mathbf{e_8}$ approaches pioneered by Tony Smith [44].

Codes and simplices associated to $\mathbb{OP}^2$ are studied in [35], including a design of 819 points. For $\mathbb{FP}^n$, over any division algebra, many of these codes come from mutually unbiased bases [45][46][47]. For $\mathbb{HP}^n$, there are $2n + 3$ unbiased bases with $(2n + 3)(n + 1)$ points. The 819 points form a distinct structure, with $819 = \sum_{i=1}^{13} i^2$ suggesting a Lorentzian vector $(0, 1, 2, \cdots, 13, 9, 30)$ in $15 + 1$ dimensions.

9 A 72 dimensional lattice

Monster moonshine begins with a vertex operator algebra based on Λ, and its trivalent vertices suggest looking at three copies of Λ in higher dimensional lattices. In the magic star [10][11], one copy of the Leech lattice in $J_3(\mathbb{O})$ extends to three copies around a triangle in the star. Now there exists an even unimodular lattice Φ in dimension 72 [48] whose minimal vectors have norm 8, making it an extremal lattice. It is constructed using three copies of Λ, along with a 6 dimensional lattice Θ known as the Barnes lattice.

Let $\alpha = (1+\sqrt{-7})/2$. Θ is a subset of vectors $v = (x, y, z)$ with components in $\mathbb{Z}[\alpha]$. Define a Hermitian form on Θ by

$$h : \Theta \times \Theta \to \mathbb{Z}[\alpha], \qquad (v, w) \mapsto \frac{1}{2}\sum_{i=1}^{3} v_i \overline{w}_i. \tag{51}$$

Θ is usually the span of the vectors $(1, 1, \alpha)$, $(0, \overline{\alpha}, \overline{\alpha})$ and $(0, 0, 2)$. Its automorphism group is $C_2 \cdot PSL_2(7)$, of order $336 = 24 \cdot 14$. We find a matrix A and its conjugate $B = I - A$, so that $(\alpha, \overline{\alpha})$ maps to (A, B), defining the three rows of

$$Q = \begin{pmatrix} I & I & A \\ 0 & B & B \\ 0 & 0 & 2I \end{pmatrix}. \tag{52}$$

The matrix A is 24-dimensional over $\mathbb{Z}$ and should satisfy

$$AGA^t = 2G, \qquad GA^tG^{-1} = B, \tag{53}$$

where G is a Gram matrix for Λ. There are 9 solutions for A modulo the automorphisms of Λ [48], but one natural choice for an extremal Φ. Now Φ is the sublattice of $(\Lambda, \Lambda, \Lambda)$ defined by Q, so that over $\mathbb{Z}[\alpha]$ it is the lattice $\Theta \otimes \Lambda$. Letting Tr denote the trace on $\mathbb{Z}[\alpha]/\mathbb{Z}$, a $\mathbb{Z}[\alpha]$ structure for Λ is $\frac{1}{7}\mathrm{Tr} \cdot h$. Then Λ contributes an $SL_2(25)$ of order 15600 to the automorphisms in 36 dimensions. The number of norm 8 vectors in Φ equals

$$31635 \cdot 196560 = 2025 \cdot 9139 \cdot |C_2 \cdot PSL_2(7)|. \tag{54}$$

Note that there is a prime factor of 37 here, which does not divide the order of $\mathbb{M}$, but the mod 37 Fibonacci numbers have a length 19 cycle. The automorphisms of Φ include the semidirect product $PSL_2(7) \cdot SL_2(25) : C_2$, of order $6400 \cdot 819$, which is four times $1209600 + 100800$, where 1209600 is the order of $2 \cdot J_2$, the symmetry of the icosian Leech lattice.

Observe that $|SL_2(25)|/(9 \cdot 240)$ equals $7 + 2/9$, where we take 9 copies of the $\mathbf{e_8}$ roots in Φ. Recall that $1080 = 9 \cdot 120$ is the order of the triple cover $3 \cdot A_6$. Now $240 = 1080 \cdot (2/9)$ reminds us of the nine copies of 24 on octonion points in the magic star. The remaining $1080 \cdot 7$ will appear in section 10, suggesting a strong link between Φ and the Monster.

The matrix (52) suggests the 72×72 circulant Koide matrix

$$K = \begin{pmatrix} I & A & B \\ B & I & A \\ A & B & I \end{pmatrix}, \tag{55}$$

where (A, I, B) is a projective splitting of idempotents with $BA = 2I$, suggesting a natural rescaling of Θ by $1/\sqrt{2}$. Now consider a map $\mathbb{Z} \to \mathbb{Z}$ mod 3 taking

$$\alpha^2 - \alpha + 2 = 0 \quad \to \quad \phi^2 - \phi - 1 = 0, \tag{56}$$

where ϕ is the golden ratio. This says that mod 3 arithmetic is closely related to the appearance of ϕ in lattice coordinates and mass phenomenology.

The Barne's matrix (52) also gives three out of four orbits for minimal vectors in the icosian Λ. Recall that $h = (-\sqrt{5} + i + j + k)/2$ generates one of the five ideals of the 600-cell. Three orbits are characterised by the vectors $(2, 0, 0)$, $(0, h, h)$ and $(\overline{h}, 1, 1)$, which looks exactly like the map $\overline{\alpha} \mapsto h$, but we need to look at something different first. These vectors generate Λ when thought of as a (left) module over the icosians [26].

The fourth orbit comes from $(1, \phi u, -\phi^{-1}\overline{u})$, where $u = (-1 + i + j + k)/2$, so that the map $u \mapsto \omega$ is analagous to $s \mapsto \alpha$. Indeed, we have $u^2 = \overline{u}$ and $u^2 + u + 1 = 0$. Then $h = u - \phi^{-1}$, so the natural reduction is $h \mapsto \omega - \phi^{-1}$, which satisfies a quadratic. Note that ω, ϕ and their conjugates solve all four quadratics $x^2 \pm x \pm 1 = 0$. This summarises the $\mathbb{Z}[\alpha]$, $\mathbb{Z}[\omega]$ and $\mathbb{Z}[\phi]$ structures on Λ in dimension 12.

Now under $\overline{\alpha} \mapsto h$ above, we would require $\omega \mapsto 0$, which is conveniently done by $J(\omega) = 0$, for $J(z) = j(z) + 744$. Clearly we need more than one copy of Φ to account for different evaluations of h.

Our musings about modular arithmetic are motivated by the prime powers p^r of quantum Hilbert spaces, with p^r acting as a coarse graining on a cubic lattice with a discrete dimension labelled by sets of p points. Going from the (root) lattices to the exponentiated group, whatever analog of a group we might use, is a process of quantization, because a root is an element of a *set* while the same root later contributes a dimension. We saw how the 8 points of a 3-cube denote the 8 dimensions of $\mathbb{O}$. With one, two or three qutrits we are in dimensions $3, 9, 27$, which combine with three qubits to give dimensions $24, 72, 216$.

How many qudits do we need for the fundamental degrees of freedom of gravity? In dimension 72, we can put 9 copies of $\mathbb{Z}^8/2$ into a 3×3 matrix to recover complex number entries using the symplectic map. The mod 27 cycle of Fibonacci numbers has length 72, which includes the length 24 cycles that start at mod 6 [18]. These lengths are expected to govern dimensions of modules in quantum gravity.

Beyond 72 is its triple $216 = 8 \cdot 81$, which is where we would construct $J_3(\mathbb{O} \otimes \mathbb{O})$. This is just sufficient to account for all copies of $\mathbb{O}$ in the magic plane. Then we have the beautiful fact that

$$\frac{1}{24} - \frac{1}{27} = \frac{1}{216} \tag{57}$$

so that $216 = 9 \cdot 24 = 8 \cdot 27$, and $1728 = 72 \cdot 24 = 64 \cdot 27$, where the 64 covers three Dirac spinors and the 27 adds three qutrits.

Finally we ask: does three copies of Φ get us close to the *group* $\mathbf{e_8}$, whose roots define $240 = 216 + 24$ dimensions. The idea is that the third tripling should somehow take us back to where we started. Recall that 24 roots are selected in the magic plane outside of the $\mathbb{O}$ components. Thus ten copies of Λ reduce to a vector $(\Phi, \Phi, \Phi, \Lambda)$ in dimension 240, and the Λ directions give the special 24 roots. The usual coordinates for the $\mathbf{a_2}$ hexagon in three dimensions are the permutations of XYZ, which were included as ribbon charges in the 24 qutrit words for the Z boson [9], along with the remaining 18 diagonal elements. Now we see that the 27 dimensions of bosonic M theory have little to do with classical spaces.

10 Prime triples and M_{24} moonshine

Qubit state spaces label vertices on a parity cube, while qutrits require cubes with midpoints on each edge, using up the coordinates $\{0, \pm 1\}$. Coarse graining a three qubit cube requires only $3 = 2^2 - 1$ points along an edge, to create eight little 3-cubes. Thus primes of the form $2^n - 1$ are the only coarse graining primes. 3 is the only prime squeezed between two prime powers. Each point on a prime power edge stands for a digit $\{0, 1, \cdots, p-1\}$ in $\mathbb{F}_p$. Digits are quantised by circulant mutually unbiased bases [45][46][47], leaving out the quantum Fourier transform. Putting a power r on every divisor n in a finite cubic lattice defines a divisor function $\sigma_r(n)$ on the target of the cubic block, so that an Eisenstein series is naturally a discrete analog of a monomial function x^r.

Three dimensions are physical. It makes sense for us to choose primes as labels for each direction in discrete space, so that three dimensions denote a prime triple (p_1, p_2, p_3). When we look at the orders of finite simple groups, we immediately think of triplets of their primes. Consider the primes dividing the order of $\mathbb{M}$, the Monster. A triple (p_1, p_2, p_3) is mapped to its polygon triple $(p_1 + 1, p_2 + 2, p_3 + 3)$, because factorization for $N \in \mathbb{N}$ occurs in chordings of a polygon with $N + 1$ sides, and there are $p + 1$ mutually unbiased bases in dimension p, including the $p \times p$ Fourier transform. Starting from the big numbers in

$$|\mathbb{M}| = 2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71, \tag{58}$$

we observe that

$$\begin{aligned} 72 \cdot 60 \cdot 48 &= 196560 + 10800, \\ 42 \cdot 32 \cdot 30 &= 4 \cdot 10080, \\ 24 \cdot 20 \cdot 18 &= 8 \cdot 1080, \\ 60 \cdot 48 \cdot 42 &= 120960 = 196560 - 7 \cdot 10800, \\ 48 \cdot 42 \cdot 32 &= 64512 = 2^{16} - 2^{10}, \end{aligned} \tag{59}$$

$$60 \cdot 30 \cdot 24 = 4 \cdot 10800,$$

where $10080 + 720 = 10800$ occurs in the squares of the modular discriminant and related Eisenstein series. Observe that

$$72 \cdot 60 \cdot 48 = (273 + 15) \cdot 720 = 196560 + 8 \cdot 720 + 7!, \tag{60}$$

where $7!$ is the number of vertices on S_7, and $8 + 7$ is the splitting $128 + 112$ for L_8. In the last section we saw that $7 \cdot 1080$ appears in the structure of the 72 dimensional lattice, and $8 \cdot 10800 = 5 \cdot 17280$, where 17280 counts holes in the L_8 lattice. We have

$$72 \cdot 60 \cdot 48 = 3 \cdot 2^{16} + 3 \cdot 14 \cdot 2^8, \tag{61}$$

where the second term is $10477 + 275$ and 10477 is prime.

The prime product $71 \cdot 59 \cdot 47 = 196883$ is the dimension of the Griess module [49][26], and $196884 = 300+98280+98304$ the dimension of the algebra, traditionally described as follows. Given two symmetric 24×24 matrices X and Y, the Griess product is $X * Y = 2(XY + YX)$, in a 300 dimensional space of matrices. In Λ, there are $98280 = 196560/2$ positive vectors v which we can use as basis vectors in dimension 98280. The remaining 98304 dimensions have basis vectors $f \otimes b$ for f a fermion spinor in dimension 4096 and b a boson in dimension 24, as occurs in the level 3 T-algebra [11] of shape

$$\begin{pmatrix} 1 & 24 & 2048 \\ 24 & 1 & 2048 \\ 2048 & 2048 & 1 \end{pmatrix}. \tag{62}$$

The action of X on b is

$$X * (f \otimes b) = f \otimes bX + \frac{1}{8}(\mathrm{Tr}X)(f \otimes b). \tag{63}$$

The Mathieu group M_{24} in $\mathbb{M}$ acts on 2048 elements of the Golay code, and on

$$4096 = 1 + 24 + 276 + 2024 + 1771, \tag{64}$$

where $1771 = 7 \cdot 11 \cdot 23$ has polytope prime dimension 48^2. The dimension 276 carries one of the rank 3 permutation representations for M_{24}, and $276 = 4{\cdot}3{\cdot}23$, with polytope dimension 480. The 2024 is required for particle mixing matrices [50], which have to be magic, requiring a space of dimension $2024 = 2048 - 24$. And $2024 = 8 \cdot 11 \cdot 23$. The Golay octads give $759 = 3 \cdot 11 \cdot 23$. Observe that these Golay prime power triples only use primes of the form $p|24$ (ie. $p = 2$ or 3) or $24/p - 1$ (for $p = 1, 2, 3$). These divisors label Niemeier lattices in umbral moonshine [51].

The primes 3, 7, 11 and 23 all satisfy $-p \equiv 1 \bmod 4$, so that $(1 + \sqrt{-p})/2$ gives an integral ring. The norm [52] of $a+b\phi$ in $\mathbb{Q}(\sqrt{5})$ is $a^2 - b^2 + ab$, indicating that the trace of ϕ equals 1 and the norm is -1, and we have $5 \equiv 1 \bmod 4$. In the integers for $\mathbb{Q}(\sqrt{-p})$, the trace is 1 whenever $-p \equiv 1 \bmod 4$, allowing the 1/2.

This extends to the usual trace and norm definitions for noncommutative and nonassociative algebras, where a trace of 1 is a projector condition for quantum states. In other words, quantum measurements naturally select integers from a quadratic field, which is real or imaginary depending on p mod 4, starting with $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{5})$, drawing hexagons and pentagons for our axioms. Since such primes often denote the diagonal length on a cube, we include $\sqrt{-71}$ for Φ. There is no unit trace for the Gaussian integers.

Adding up the number of words in the code and cocode, omitting the zero vectors, we obtain

$$2 \cdot 4095 = 8190, \tag{65}$$

where $8190 \cdot 24 = 196560$ in Λ. Congruence classes of norm 8 vectors in Λ come in sets of 48, called *crosses*, such that the stabiliser of a cross in $2 \cdot \mathrm{Co}_1 = \mathrm{Aut}(\Lambda)$ is the semidirect product $2^{12} : M_{24}$. There are $48 \cdot 45^3 \cdot 7 \cdot 13$ norm 8 vectors. Thinking of $\mathbb{OP}^2$, 196883 also equals $3 \cdot 2^{16} + 275$, where 275 is the dimension of the 256 spinor T-algebra at level 2, of shape

$$\begin{pmatrix} 1 & 16 & 128 \\ 16 & 1 & 128 \\ 128 & 128 & 1 \end{pmatrix}. \tag{66}$$

A level 9 algebra should be studied for the lattice Φ. Observe the interesting ring homomorphisms that enter dimensional reduction. Let $R = \mathbb{Z}/3\mathbb{Z}$. If we start with $\mathbb{Z}[\alpha]$, taking mod 3 introduces $R[\phi]$, the 9 element ring using the values (20). In $R[\phi]$ with ϕ indeterminate, we recover formal Lucas numbers $L_n = \phi^n + (-1/\phi)^n$, but the Fibonacci numbers satisfy

$$5F_n = 2L_{n+1} - L_n, \tag{67}$$

requiring a fractional ideal in $\mathbb{Z}[\phi]$.

11 Monster moonshine

It seems clear that this information theoretic approach brings clarity to the physical meaning of CFSG and moonshine coincidences. In quantum gravity, the observer measures, and the observer's frame of mind depends on cosmological parameters. BTZ black holes are directly relevant to gravity in a holographic approach, where we restrict our attention to true two dimensional CFTs. The j-invariant appears in Witten's Monster CFT [4] at $c = 24$. In categorical condensed matter physics, we use knots and other diagrams, axiomatically, for quantum computation. Manifolds are a derived concept. We want to employ the categorical polytopes within cubic lattices, already well established in particle physics, to the task of deriving any modular form from first principles.

Combining a Dirac 4-dit and qutrit in three dimensions gives a 1728 vertex cube, and the normalisation of the j-invariant. Let us write the first few terms of (24) as

$$\begin{aligned} 196560 &= 240 \cdot 819 \\ 16773120 &= 5 \cdot 819 \cdot 2^{12} \\ 398034000 &= 25 \cdot 81 \cdot 240 \cdot 819 \\ 4629381120 &= 5 \cdot 276 \cdot 819 \cdot 2^{12} \\ 34417656000 &= 25 \cdot 7004 \cdot 240 \cdot 819 \\ 187489935360 &= 5 \cdot 11178 \cdot 819 \cdot 2^{12} \\ 814879774800 &= 5 \cdot 829141 \cdot 240 \cdot 819 \\ 2975551488000 &= 5 \cdot 177400 \cdot 819 \cdot 2^{12}, \end{aligned} \tag{68}$$

where 276, 7004, 11178, 829141 and 177400 all have three factors. The first coefficient of the j-invariant (16) is

$$c(1) = 196884 = 196560 + 324 = 3 \cdot 2^{16} + 276, \tag{69}$$

recalling the 15-spheres for $\mathbb{OP}^2$. Frenkel's study of the Kac-Moody algebra associated to the Lorentzian Leech lattice [53] gives a bound of 324 on multiplicities of roots of norm -2, where 324 is $p_{24}(2)$, the number of partitions in 24 colors of 2. It includes $276 = \binom{24}{2}$ when the colors are distinct.

The second coefficient of j is

$$c(2) = 21493760 = 5 \cdot 819 \cdot 2^{11} + 25 \cdot 2^{19}, \tag{70}$$

noting the appearance of the norm 6 number 16773120/2 from f_Λ, along with the 2^{19} spinor in the T-algebra of shape

$$\begin{pmatrix} 1 & 40 & 2^{19} \\ 40 & 1 & 2^{19} \\ 2^{19} & 2^{19} & 1 \end{pmatrix} \tag{71}$$

at level 5, which is two levels beyond the Leech lattice. The dimension of this algebra is $24 + 5 \cdot 209719$, which equals $11 \cdot 13 \cdot 7333$, introducing new primes.

At level 6 the spinor dimension becomes $2^{32} = 65536^2$, and for the 72 dimensional lattice Φ we go up to level 9, with $2^{72} = (2^{24})^3$. The number 2^{24} counts the elements in the power set on a 24 element set. Now observe that

$$2^{24} - 16773120 = 2^{12} = 4096, \tag{72}$$

recovering the spinor dimension at level 3. Then

$$(2^{24})^2 = (2^{24} + 2^{13}) \cdot 4095^2 + 4096 \cdot (3 \cdot 4095 + 1), \tag{73}$$

making heavy use of the $4095 = 196560/48 = 5 \cdot 819$, which counted words in the Golay code. Finally, for fun, note that $259200 = (8 + 7) \cdot 17280$ is $360 \cdot 6!$, traditionally the number of seconds in three days, or years in ten Earth precession cycles. Time is dimensionless when measured as a ratio.

Differences in powers of 2 of the form

$$D_n(\delta) = 2^{n+\delta} - 2^n \tag{74}$$

have a fixed set of prime factors at sufficiently large n. For example, at $\delta = 20$ we have the primes $2, 3, 5, 11, 31, 41$. All primes for $\delta \in \{4, 12, 20\}$ (associated to levels $1, 3$ and 5) divide $|\mathbb{M}|$. The T-algebra bosonic components at these levels sum to

$$72 = 8 + 24 + 40, \tag{75}$$

justifying (70).

Ignoring the S_3 permutations, Wilson's $\mathbb{O}^3$ version of the automorphism group $2 \cdot \mathrm{Co}_1$ for Λ is generated by the 3×3 matrices [27]

$$-\frac{1}{2}\begin{pmatrix} 0 & \overline{\alpha} & \overline{\alpha} \\ \alpha & 1 & -1 \\ \alpha & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix}, \quad \frac{1}{\sqrt{2}}(1-i)I_3, \quad \frac{1}{\sqrt{2}}(1+e)I_3, \tag{76}$$

where I_3 is the identity, e is one of the six units other than 1 and i, and α appears above in the structure of Φ. Recall A and B in (52). The vector $(0, B, B)$ maps to $(0, \overline{\alpha}, \overline{\alpha})$ in the first row of the first matrix. But $(A, 1, 1)$ is replaced by $(A, 1, -1)$, which is possible in Θ with $(A, 1, 1) - (0, 0, 2)$. Thus the first matrix defines Θ for Φ.

Nine copies of L_8 make $SL_3(\mathbb{C})$. Four copies make $SL_2(\mathbb{C})$. The primes 2 and 3 underlie a great deal indeed, just as Francis Brown discovered with multiple zeta values, which extend the integral arguments of the Riemann zeta function to partitions.

Having put cubes and permutohedra everywhere possible in the reduced spaces, we are in a position to talk about (in general, non symmetric) operads built from these polytopes, including the little d-cubes operad in any dimension, the permutohedra themselves, and the Solomon parity cubes derived from the permutohedra. Combining compositions from different operads, we build higher dimensional operads, aiming for a 3-ordinal description of moonshine. Recall that the permutohedron vertices are binary rooted trees, and S_4 is mapped here to the icosians in $\mathbb{H}$. Every point is canonically labeled by both a diagram and algebraic counterparts, as motivic structures should.

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