Evidential distance measure in complex belief function theory

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***Abstract*—In this paper, an evidential distance measure is proposed which can measure the difference or dissimilarity between complex basic belief assignments (CBBAs), in which the CBBAs are composed of complex numbers. When the CBBAs are degenerated from complex numbers to real numbers, the proposed distance will degrade into the Jousselme et al.'s distance. Therefore, the proposed distance provides a promising way to measure the differences between evidences in a more general framework of complex plane space.**

***Index Terms*—Evidential distance measure, Complex belief function, Complex basic belief assignments, Complex number.**

I. The complex basic belief assignment

A generalization of Dempster–Shafer evidence (GDSE) theory is presented recently, in which a new concept of complex belief function is defined based on the complex numbers [1].

Let Ω be a set of mutually exclusive and collective non-empty events, defined by

$$\Omega = \{e_1, e_2, \ldots, e_i, \ldots, e_n\}, \tag{1}$$

where Ω represents a frame of discernment.

The power set of Ω is denoted by 2^Ω, in which

$$\begin{aligned} 2^\Omega = \{\emptyset, \{e_1\}, \{e_2\}, \ldots, \{e_n\}, \{e_1, e_2\}, \ldots, \{e_1, \\ e_2, \ldots, e_i\}, \ldots, \Omega\}, \end{aligned} \tag{2}$$

and $\emptyset$ is an empty set.

Definition 1: (Complex mass function)

A complex mass function $\mathbb{M}$ in the frame of discernment Ω is modeled as a complex number, which is represented as a mapping from 2^Ω to $\mathbb{C}$, defined by

$$\mathbb{M}: \quad 2^\Omega \to \mathbb{C}, \tag{3}$$

satisfying the following conditions,

$$\begin{aligned} \mathbb{M}(\emptyset) &= 0, \\ \mathbb{M}(A) &= \mathbf{m}(A)e^{i\theta(A)}, \quad A \in 2^\Omega \\ \sum_{A \in 2^\Omega} \mathbb{M}(A) &= 1, \end{aligned} \tag{4}$$

where $i = \sqrt{-1}$; $\mathbf{m}(A) \in [0,1]$ representing the magnitude of the complex mass function $\mathbb{M}(A)$; $\theta(A) \in [-\pi, \pi]$ denoting a phase term.

In Eq. (4), $\mathbb{M}(A)$ can also be expressed in the "rectangular" form or "Cartesian" form, denoted by

$$\mathbb{M}(A) = x + yi, \quad A \in 2^\Omega \tag{5}$$

with

$$\sqrt{x^2 + y^2} \in [0, 1]. \tag{6}$$

By using the Euler's relation, the magnitude and phase of the complex mass function $\mathbb{M}(A)$ can be expressed as

$$\mathbf{m}(A) = \sqrt{x^2 + y^2}, \text{ and } \theta(A) = \arctan(\frac{y}{x}), \tag{7}$$

where $x = \mathbf{m}(A)\cos(\theta(A))$ and $y = \mathbf{m}(A)\sin(\theta(A))$.

The square of the absolute value for $\mathbb{M}(A)$ is defined by

$$|\mathbb{M}(A)|^2 = \mathbb{M}(A)\overline{\mathbb{M}}(A) = x^2 + y^2, \tag{8}$$

where $\overline{\mathbb{M}}(A)$ is the complex conjugate of $\mathbb{M}(A)$, such that $\overline{\mathbb{M}}(A) = x - yi$.

These relationships can be then obtained as

$$\mathbf{m}(A) = |\mathbb{M}(A)|, \text{ and } \theta(A) = \angle\mathbb{M}(A), \tag{9}$$

where if $\mathbb{M}(A)$ is a real number (i.e., $y = 0$), then $\mathbf{m}(A) = |x|$.

If $|\mathbb{M}(A)|$ ($A \in 2^\Omega$) is greater than zero, A is called a focal element of the complex mass function. The value of $|\mathbb{M}(A)|$ represents how strongly the evidence supports A.

The complex mass function $\mathbb{M}$ modeled as a complex number in the generalized Dempster–Shafer (GDS) evidence theory can also be called a complex basic belief assignment (CBBA). When $\mathbb{M}(A)$ degrades into a real number, a CBBA will degrades into a BBA.

Definition 2: (Complex belief function)

Let Ω be a frame of discernment, and $A \in 2^\Omega$. The complex belief function of A, denoted as $Bel_c(A)$ is defined by

$$Bel_c(A) = \sum_{B \subseteq A} \mathbb{M}(B). \tag{10}$$

Definition 3: (Complex plausibility function)

Let Ω be a frame of discernment, and $A \in 2^\Omega$. The complex plausibility function of A, denoted as $Pl_c(A)$ is defined by

$$Pl_c(A) = \sum_{B \cap A \neq \emptyset} \mathbb{M}(B). \tag{11}$$

II. A new distance measure between complex basic belief assignments

In this section, a new evidential distance measure for complex basic belief assignments is proposed.

Definition 4: (Evidential distance measure between CBBAs).

Let $\mathbb{M}_1$ and $\mathbb{M}_2$ be two CBBAs on the frame of discernment Ω, where A and B are the hypotheses of CBBAs $\mathbb{M}_1$ and $\mathbb{M}_2$, respectively. The evidential distance measure between the

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CBBAs $\mathbb{M}_1$ and $\mathbb{M}_2$, denoted as $d_{CBBA}(\mathbb{M}_1, \mathbb{M}_2)$ is defined by

$$d_{CBBA}(\mathbb{M}_1, \mathbb{M}_2) = \sqrt{\frac{|(\overrightarrow{\mathbb{M}}_1 - \overrightarrow{\mathbb{M}}_2)^T \underline{\underline{\mathbb{D}}}(\overrightarrow{\mathbb{M}}_1 - \overrightarrow{\mathbb{M}}_2)|}{\sum\limits_{A \subseteq \Omega} |\mathbb{M}_1(A)| + \sum\limits_{B \subseteq \Omega} |\mathbb{M}_2(B)|}}, \quad (12)$$

where $\overrightarrow{\mathbb{M}}$ is the vector of CBBA $\mathbb{M}$; $(\overrightarrow{\mathbb{M}}_1 - \overrightarrow{\mathbb{M}}_2)^T$ is the transposition of $(\overrightarrow{\mathbb{M}}_1 - \overrightarrow{\mathbb{M}}_2)$; $|\cdot|$ denotes the absolute function; $\underline{\underline{\mathbb{D}}}$ represents a $2^n \times 2^n$ matrix which has the following elements

$$\underline{\underline{\mathbb{D}}}(A, B) = \frac{|A \cap B|}{|A \cup B|}. \quad (13)$$

In Eq. (12), $\sum\limits_{A \subseteq \Omega} |\mathbb{M}_1(A)| + \sum\limits_{B \subseteq \Omega} |\mathbb{M}_2(B)|$ is required to normalize d_{CBBA}.

For Eq. (12), it can be expressed by another form,

$$d_{CBBA}(\mathbb{M}_1, \mathbb{M}_2) = \sqrt{\frac{\|\overrightarrow{\mathbb{M}}_1\|^2 + \|\overrightarrow{\mathbb{M}}_2\|^2 - 2|\langle \overrightarrow{\mathbb{M}}_1, \overrightarrow{\mathbb{M}}_2 \rangle|}{\sum\limits_{A_i \in 2^\Omega} |\mathbb{M}_1(A_i)| + \sum\limits_{A_j \in 2^\Omega} |\mathbb{M}_2(A_j)|}}, \quad (14)$$

where $|\langle \overrightarrow{\mathbb{M}}_1, \overrightarrow{\mathbb{M}}_2 \rangle|$ represents the scalar product, which is defined as

$$|\langle \overrightarrow{\mathbb{M}}_1, \overrightarrow{\mathbb{M}}_2 \rangle| = \left| \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} \mathbb{M}_1(A_i) \overline{\mathbb{M}}_2(A_j) \frac{|A_i \cap A_j|}{|A_i \cup A_j|} \right|, \quad (15)$$

$\overline{\mathbb{M}}_2(A_j)$ is the complex conjugate of $\mathbb{M}_2(A_j)$, and $\|\overrightarrow{\mathbb{M}}\|^2$ is the square norm of $\overrightarrow{\mathbb{M}}$, defined by

$$\begin{aligned} \|\overrightarrow{\mathbb{M}}\|^2 &= |\langle \overrightarrow{\mathbb{M}}, \overrightarrow{\mathbb{M}} \rangle| \\ &= \left| \sum_{i=1}^{2^n} \sum_{j=1}^{2^n} \mathbb{M}(A_i) \overline{\mathbb{M}}(A_j) \frac{|A_i \cap A_j|}{|A_i \cup A_j|} \right|. \end{aligned} \quad (16)$$

It is obvious that when the CBBAs are degraded from complex numbers to real numbers, the proposed distance measure degrades into the Jousselme et al.'s distance measure [2].

The properties of the proposed distance measure can be summarized as

Property 1: Let $\mathbb{M}_1$, $\mathbb{M}_2$ and $\mathbb{M}_3$ be arbitrary three CBBAs, then

P2.1 Non-negativity: $d_{CBBA}(\mathbb{M}_1, \mathbb{M}_2) \geq 0$.

P2.2 Non-degeneracy: $d_{CBBA}(\mathbb{M}_1, \mathbb{M}_2) = 0$ if and only if $\mathbb{M}_1 = \mathbb{M}_2$.

P2.3 Symmetry: $d_{CBBA}(\mathbb{M}_1, \mathbb{M}_2) = d_{CBBA}(\mathbb{M}_2, \mathbb{M}_1)$.

P2.4 Triangle inequality: $d_{CBBA}(\mathbb{M}_1, \mathbb{M}_3) \leq d_{CBBA}(\mathbb{M}_1, \mathbb{M}_2) + d_{CBBA}(\mathbb{M}_2, \mathbb{M}_3)$.

References

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[2] A.-L. Jousselme, D. Grenier, and É. Bossé, "A new distance between two bodies of evidence," *Information Fusion*, vol. 2, no. 2, pp. 91–101, 2001.