The Inconsistency of Arithmetic

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Abstract. Based on a strengthened form of the strong Goldbach conjecture, this paper presents an antinomy within the Peano arithmetic (PA). We derive two contradictory statements by using the same main instrument as in the proof [2] of the conjecture, that is, a structuring of the natural numbers starting from 3.

Notations. Let $\mathbb{N}$ denote the natural numbers starting from 1 and let $\mathbb{P}_3$ denote the prime numbers starting from 3.

Theorem. *The Peano arithmetic (PA) is inconsistent.*

Proof. We define the set

$$S_g := \{ (pk, mk, qk) \mid k, m \in \mathbb{N};\ p, q \in \mathbb{P}_3,\ p < q;\ m = (p + q) / 2 \}$$

and we consider the two cases

(G) The numbers m in the components mk take all integer values $x \geq 4$.

$\neg$(G) The numbers m in the components mk do not take all integer values $x \geq 4$.

Furthermore, we consider the sum

$$sum_g := \sum_{k \in \mathbb{N}} \ \sum_{\substack{p,q \in \mathbb{P}_3 \\ p < q \\ m=(p+q)/2}} ((pk + qk) / 2 - mk)$$

Trivially, we have $sum_g = 0$ since each term $(pk + qk) / 2 - mk$ is zero. As $sum_g = 0$ holds regardless of whether (G) or $\neg$(G) applies, we can state

(I) ((G) => $sum_g = 0$) AND ($\neg$(G) => $sum_g = 0$).

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[2] http://vixra.org/abs/1702.0300

On the other hand, there is the following relationship between sum_g in case (G) and sum_g in case ¬(G).

The case ¬(G) means that there is at least one n ≥ 4 such that nk is different from all the mk for each k ≥ 1. All pairs of odd primes, that determine the numbers m, are used in S_g.

The positive part of sum_g is the same in both cases (G) and ¬(G) since for each k ≥ 1 such an nk given by the case ¬(G) is contained in this part as some pk when n is prime, as some pk' when n is composite and not a power of 2, or as (3 + 5)k' / 2 when n is a power of 2; $p \in \mathbb{P}_3$; $k, k' \in \mathbb{N}$.

While the positive summands of sum_g are the same in both cases, in the negative part for each k ≥ 1 at least the summand −nk does not exist in case ¬(G) whereas it exists in case (G). Therefore, we obtain

(((G) => $sum_g = 0$) => (¬(G) => $sum_g \neq 0$))

OR

((¬(G) => $sum_g = 0$) => ((G) => $sum_g \neq 0$))

<=>

(((G) AND $sum_g \neq 0$) OR (¬(G) => $sum_g \neq 0$))

OR

((¬(G) AND $sum_g \neq 0$) OR ((G) => $sum_g \neq 0$)).

According to (I), the parts ((G) => $sum_g \neq 0$) and (¬(G) => $sum_g \neq 0$) are false. So, we get

(II) ((G) AND $sum_g \neq 0$) OR (¬(G) AND $sum_g \neq 0$). □

The proof uses a strengthened form of the strong Goldbach conjecture:

Strengthened strong Goldbach conjecture (SSGB): *Every even integer greater than 6 can be expressed as the sum of two different primes.*

SSGB is equivalent to saying that all integers x ≥ 4 appear as m in a component mk of S_g. Therefore, SSGB is equivalent to the case (G) and the negation ¬SSGB is equivalent to the case ¬(G).