Theorem 1 Let G be a group of order $p^n m$, where p is prime and $p \nmid m$. Suppose that G has a normal subgroup P of order p^n. Prove that $\theta(P) \subset P$ for every automorphism θ of G.

Proof.
Let θ be an automorphism of G. Since P is a normal subgroup of G and $\theta(P)$ is a subgroup of G, $P\theta(P)$ is a subgroup of G. Moreover $|\theta(P)| = |P|$. Since $P \cap \theta(P)$ is a subgroup of P, $|P \cap \theta(P)|$ divides $|P|$. So $|P \cap \theta(P)| = p^k$ where $0 \leqq k \leqq n$. Thus

$$|P\theta(P)| = \frac{|P||\theta(P)|}{|P \cap \theta(P)|} = \frac{p^n p^n}{|P \cap \theta(P)|}.$$

To claim $|P \cap \theta(P)| = p^n$. If not, then $|P \cap \theta(P)| = p^k$ where $k < n$. Thus

$$|P\theta(P)| = \frac{p^n p^n}{p^k} = p^{2n-k},$$

so is divisible by p^{n+1} since $2n - k \geqq n + 1$. But $|P\theta(P)|$ divides $|G|$; that is $|P\theta(P)|$ divides $p^n m$. So $p^{n+1} | p^n m$ and thus $p|m$ which is a contradiction. Since $|P \cap \theta(P)| = p^n$, $P \cap \theta(P) = \theta(P)$ and hence $\theta(P) \subset P \cap \theta(P) \subset P$.

Theorem 2 If N is a normal subgroup of G and $M \subset N$ is a characteristic subgroup of N. Then M is a normal subgroup of G.

Proof.
Let $a \in G$, $n \in N$. Since N is a normal subgroup of G, $a^{-1}na \in N$. Define $\varphi : N \to N$ by $\varphi(n) = a^{-1}na$. Thus φ is an automorphism of N. Let $m \in M$. Since $M \subset N, m \in N$. So $a^{-1}ma = \varphi(m) \in \varphi(M) \subset M$. To conclude $a^{-1}ma \in M$.

Theorem 3 $V = \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$ is normal in S_4.

Proof.
$|A_4| = 2^2 3$ where 2 is prime and $2 \nmid 3$. Also, V is a normal subgroup of A_4 of order 2^2. By Theorem 1, $V \subset A_4$ is a characteristic subgroup of A_4. Since A_4 is a normal subgroup of S_4 and $V \subset A_4$ is a characteristic subgroup of A_4, V is a normal subgroup of S_4 by Theorem 2.