ASYMPTOTIC CLOSED-FORM N^{th} ZERO FORMULA FOR RIEMANN ZETA FUNCTION

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ABSTRACT

Assuming the Riemann Hypothesis to be true, we propose an asymptotic and closed-form formula to find the imaginary part for non-trivial zeros of the Riemann Zeta Function.

***K*eywords** Riemann Hypothesis · Riemann Zeta Function · Non-trivial Zeros · N^{th} Zero Formula

1 Introduction

For a long time ago the non-trivial zeros of Riemann Zeta Function has been focus of intense investigation, actually considered the most important unsolved problem in pure mathematics.

Significant efforts to solve the problem have been done in the past, recently notable equation published by Simon Plouffe [1] returns non-trivial zeros with increasing accuracy at n tends to infinity, as an asymptotic formula.

Inspire by his work we start our present research, finding a closed-form version.

2 On the values of $\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+in\right)$

As stated in his paper, a formula for approximate imaginary part of every non-trivial zeros is:

$$2\pi\frac{n-\frac{11}{8}}{W\left(\frac{n-\frac{11}{8}}{e}\right)} \tag{1}$$

Trough Puiseux series expansion at $n=\infty$ from equation (1) we have:

$(15/16)\pi(-1+\log n)^3((176(-1+\log n)(480+60\log^8 n^2+30\log^2(-113+26n)-60\log(-29+28n)-\log^7 n(120+497n)+5\log^4(-121+568n)+\log^6 n(929+1555n)+10\log^3(233-81n+84n^2)-2\log^5(-24+1315n+905n^2)))/(60-60\log^7 n^2-360\log(1+n)+120\log^2(5+18n)+\log^6 n(132+425n)-30\log^3(16+115n+30n^2)+10\log^4(14+261n+165n^2)-\log^5(12+925n+1230n^2))^2-(605(115920+720\log^{14} n^2-5040\log(-135+152n)-12\log^{13} n(120+917n)-360\log^2(4397+4298n)+12\log^{12} n(1877+5791n)-3\log^{11} n(51820+65117n)+\log^{10}(-432+587643n-83325n^2)+360\log^3(7215+27145n+9422n^2)-60\log^4(60009+344684n+225528n^2)+24\log^7(24693+280414n+954282n^2)+24\log^5(138147+1043230n+1089935n^2)+\log^9(10908-1116117n+2726096n^2)-2\log^8(55479+86884n+5336344n^2)-4\log^6(455752+4495554n+7668825n^2)))/(n(-60+60\log^7 n^2+360\log(1+n)-120\log^2(5+18n)-\log^6 n(132+425n)+30\log^3(16+115n+30n^2)-10\log^4(14+261n+165n^2)+\log^5(12+925n+1230n^2))^3)-(128n(-1+\log n))/(60-60\log(6+5n)+300\log^2(-2+3n+2n^2)-30\log^3(-16-35n+40n^2+20n^3)+10\log^4(-14-63n-63n^2+90n^3+30n^4)+\log^6 n(-12-15n-20n^2-30n^3+60n^4)+\log^5(12+155n+170n^2+210n^3-360n^4-60n^5)))$

This is a closed-form expression that contains only π, $\log$ and simple exponentiation. Thus now, we can find the zeros of Riemann zeta function efficiently computable at n sufficiently large.

Another mathematical expression, also related to equation (1), for non-trivial zeros with improving precision is the following one:

$$\frac{2\pi\left(n-\frac{11}{\frac{x11\pi}{\log\left(xe^{1+x\pi}\pi\right)}}\right)}{W\left(\frac{n-\frac{11}{\frac{x11\pi}{\log\left(xe^{1+x\pi}\pi\right)}}}{e}\right)} \tag{2}$$

Where $x \approx 0.133$

Expressed in Mathematica Language as:

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2*Pi*(100 - 11/((0.133*11*Pi)/
Log[0.133*E^(1 + 0.133*Pi)*Pi]))/
ProductLog[(100 - 11/((0.133*11*Pi)/
Log[0.133*E^(1 + 0.133*Pi)*Pi]))/E]
```

3 Conclusions

A closed-form and mathematical expressions to obtain every approximated non-trivial zeros of the Riemann zeta function are given.

Attempting to fix x variable of equation (2), it is possible to get a final formula that allows an arbitrary precision calculation for every non-trivial zeros.

Our formula can be interpreted as the first one for generating primes. No such formula was known until now.

References

[1] PLOUFFE, S. On the values of the functions $\frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+in\right)$ and $\frac{1}{\pi}\arg\Gamma\left(\frac{1}{4}+\frac{in}{2}\right)$. *viXra* **1408.0180** (2013).