Vector Subspaces

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Abstract

We endeavor to show certain contradictions in the theory of linear vector spaces.

Introduction

The article endeavors to bring out contradictions in the theory of linear vector spaces.

Calculations

Consider a linear vector space[1] V,dim $V = n$ and a subspace[2] $W \subset V$,dim $W = k$

V contains n linearly independent basic vectors, $e_1, e_2, e_3 \ldots . e_n$. W contains the basic vectors$e_{k+1}, e_{k+2}, e_{k+3} \ldots . e_n$. $V - W$ contains n-k linearly independent vectors: $e_1, e_2, e_3 \ldots . e_{n-k}$.

We consider a case where

$$\alpha = \sum_i c_i e_i \in W; e_i \in V - W; \alpha \neq 0 \;\; (1)$$

Since $\alpha \in W$we may write,

$$\alpha = \sum_j d_j e_j; e_j \in W \;\; (2)$$

From(1) and (2)

$$\sum_i c_i e_i = \sum_j d_j e_j \Rightarrow \sum_i c_i e_i - \sum_j d_j e_j = 0 \; (3)$$

But the vectors $e_i$and$e_j$form a linearly independent set. Therefore

$$c_i = d_j = 0 \Rightarrow \alpha = 0$$

But right at the outset we have assumed that $\alpha \neq 0$

Therefore we simply cannot have

$$\alpha = \sum_i c_i e_i \in W$$

if $e_i \in V - W$

Thus we have the following theorem

Theorem 1

If $e_i \in V - W$

$$\alpha = \sum_i c_i e_i \in V - W$$

Theorem 2

If $a \in W$ and $b \in W - V$ then $a + b \in W - V [a, b \neq 0]$

Proof: If possible let $c = a + b \in W$

$$b = c - a \quad (4)$$

Since $c \in W$, $a \in W$ the RHS of belongs to W while $b \in W - V$. Therefore (4) is not possible

Therefore $c = a + b \notin W \Rightarrow c = a + b \in V - W$

We consider $\beta \in V - W$ and $\alpha \in W$

By theorem 2,

$$\beta' = \beta + \alpha \in V - W \ (6)$$

$$\beta' - \beta = \alpha \in W$$

$$\beta' - \beta \in W \quad (7)$$

But by theorem 1

$$\beta' - \beta \in V - W \ (8)$$

Now β and β' cannot be linearly independent. If they were so, then, $\beta = \lambda\beta'$

$$\alpha = \beta - \beta' = \lambda\beta' - \beta' = (\lambda - 1)\beta'$$

$$\beta' = \frac{1}{\lambda - 1}\alpha$$

$$\Rightarrow \beta' \in W \ (9)$$

which is not true: equations (6) and (9)contradict each other

Therefore $\beta$and β' are linearly independent we expand this linearly independent set to'n' vectors which obviously span V, $\beta$and β' belonging to $V - W$.We have $\beta$and β'as tow of the n-k linearly dependent vectors [from the basis of V]that lie in V-From theorem2 a superposition of n-k vectors cannot give rise to any vector in W.That contradicts (7)

Alternative Treatment

Next we go for an alternative treatment

We take $e \in V - W$ and N vectors$y_i \in W; i = 1,2,3 \dots N \gg n$. All$y_i$ are not linearly independent.

We form the sums

$$\alpha_i = e + y_i$$

We consider the equation

$$\sum_i c_i \alpha_i = 0 \ (10)$$

$$\sum_i c_i (e + y_i) = 0$$

$$\Rightarrow e \sum_i c_i = -\sum_i c_i y_i \quad (11)$$

The right side of (11)belongs to Wwhile the left side belongs to$V - W$

[To note that if $e \in V - W$ then scalar$\times$ $e \in V - W$. Indeed if scalar$\times$ $e \in W$then $e \in W$[Wbeing a subspace]But $e \notin W$]

This is not possible unless is not possible unless all $c_i = 0$. That again makes the N,α_i from (10) linearly independent. But N>>n=dimension of V

Conclusions

As claimed at the outset, there are contradictions in the theory of the linear vector spaces. A restructuring of the subject could be necessary

References

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