Vector Subspaces

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Abstract

We endeavor to show certain contradictions in the theory of linear vector spaces.

Introduction

The article endeavors to bring out contradictions in the theory of linear vector spaces.

Calculations

Consider a linear vector space\(^1\) \(V, \dim V = n\) and a subspace\(^2\) \(W \subset V, \dim W = k\)

\(V\) contains \(n\) linearly independent basic vectors, \(e_1, e_2, e_3 \ldots e_n\). \(W\) contains the basic vectors \(e_{k+1}, e_{k+2}, e_{k+3} \ldots e_n\). \(V - W\) contains \(n-k\) linearly independent vectors: \(e_1, e_2, e_3 \ldots e_{n-k}\).

We consider a case where

\[
\alpha = \sum c_i e_i \in W; e_i \in V - W; \alpha \neq 0 \quad (1)
\]

Since \(\alpha \in W\) we may write,

\[
\alpha = \sum d_j e_j; e_j \in W \quad (2)
\]

From (1) and (2)

\[
\sum c_i e_i = \sum d_j e_j \Rightarrow \sum c_i e_i - \sum d_j e_j = 0 \quad (3)
\]

But the vectors \(e_i\) and \(e_j\) form a linearly independent set. Therefore

\[
c_i = d_j = 0 \Rightarrow \alpha = 0
\]

But right at the outset we have assumed that \(\alpha \neq 0\)

Therefore we simply cannot have

\[
\alpha = \sum c_i e_i \in W
\]
if $e_i \in V - W$

Thus we have the following theorem

**Theorem 1**

If $e_i \in V - W$

$$\alpha = \sum_i c_i e_i \in V - W$$

**Theorem 2**

If $a \in W$ and $b \in W - V$ then $a + b \in W - V [a, b \neq 0]$

Proof: If possible let $c = a + b \in W$

$$b = c - a \quad (4)$$

Since $c \in W$, $a \in W$ the RHS of $b$ belongs to $W$ while $b \in W - V$. Therefore (4) is not possible

Therefore $c = a + b \notin W \Rightarrow c = a + b \in V - W$

We consider $\beta \in V - W$ and $\alpha \in W$

By theorem 2,

$$\beta' = \beta + \alpha \in V - W \quad (6)$$

$$\beta' - \beta = \alpha \in W$$

$$\beta' - \beta \in W \quad (7)$$

But by theorem 1

$$\beta' - \beta \in V - W \quad (8)$$

Now $\beta$ and $\beta'$ cannot be linearly independent. If they were so, then $\beta = \lambda \beta'$

$$\alpha = \beta - \beta' = \lambda \beta' - \beta' = (\lambda - 1) \beta'$$

$$\beta' = \frac{1}{\lambda - 1} \alpha$$

$$\Rightarrow \beta' \in W \quad (9)$$

which is not true: equations (6) and (9) contradict each other
Therefore $\beta$ and $\beta'$ are linearly independent we expand this linearly independent set to 'n' vectors which obviously span $V$, $\beta$ and $\beta'$ belonging to $V-W$. We have $\beta$ and $\beta'$ as tow of the n-k linearly dependent vectors [from the basis of $V$] that lie in $V$-From theorem 2 a superposition of n-k vectors cannot give rise to any vector in $W$. That contradicts (7)

**Alternative Treatment**

Next we go for an alternative treatment

We take $e \in V-W$ and $N$ vectors $y_i \in W; i = 1, 2, 3 \ldots N \gg n$. All $y_i$ are not linearly independent.

We form the sums

$$\alpha_i = e + y_i$$

We consider the equation

$$\sum_i c_i \alpha_i = 0 \quad (10)$$

$$\sum_i c_i (e + y_i) = 0$$

$$\Rightarrow e \sum_i c_i = - \sum_i c_i y_i \quad (11)$$

The right side of (11) belongs to $W$ while the left side belongs to $V-W$

[To note that if $e \in V-W$ then scalar $\times e \in V-W$. Indeed if scalar $\times e \in W$ then $e \in W$ [W being a subspace] But $e \notin W$]

This is not possible unless is not possible unless all $c_i = 0$. That again makes the $N, \alpha_i$ from (10) linearly independent. But $N>>n=\text{dimension of } V$

**Conclusions**

As claimed at the outset, there are contradictions in the theory of the linear vector spaces. A restructuring of the subject could be necessary

**References**

