Some Diophantine Approximation Problems Equivalent to the Lonely Runner Conjecture

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Abstract

We prove the Lonely Runner Conjecture (LRC) is equivalent to a set of Diophantine approximation problems.

1 Introduction and Lemmas

The Lonely Runner Conjecture (LRC) asks the following question: Suppose that k runners with speeds $s_1 < s_2 < \ldots < s_k$ begin running down a circular track from the same starting line. Does each runner become lonely at some time, i.e., seperated by a distance of at least $\frac{1}{k}$ from each of the other $k-1$ runners? This conjecture is intriguing because it can be simply stated, yet is surprisingly difficult to prove and has rich connections to several branches of mathematics including graph theory, Diophantine approximations and view-obstruction problems. The LRC has continued to challenge mathematicians since it was first proposed in 1967 by J.M. Willis and has both delighted and puzzled the many mathematicians who have attacked the problem. We'll start by proving a few needed lemmas. These lemmas will allow us to show that the LRC is equivalent to a set of Diophantine approximation problems

Lemma 1. Assume $s_1 < s_2 < s_3 < \ldots s_k$. Then runner 1 becomes lonely if and only if there exists $n_1, n_2, \ldots n_{k-1} \in \mathbb{N}$ such that

$$\bigcap_{i=2}^{k} \left[\frac{n_{i-1} + \frac{1}{k}}{s_i - s_1}, \frac{n_{i-1} + \frac{k-1}{k}}{s_i - s_1} \right] \neq \emptyset \tag{1}$$

Proof. ($\Rightarrow$). Assume that runner 1 becomes lonely at some time T. Then each of the runners i are between a distance $\frac{1}{k}$ and $\frac{k-1}{k}$ ahead of runner 1 at time T. This means that each have been running between a time $\frac{n_{i-1}+\frac{1}{k}}{s_i - s_j}$ and $\frac{n_{i-1}+\frac{k-1}{k}}{s_i - s_j}$ where $n_{i-1} \in \mathbb{N}$. Hence the intersection

$$\bigcap_{i=2}^{k}\left[\frac{n_{i-1}+\frac{1}{k}}{s_i-s_1},\frac{n_{i-1}+\frac{k-1}{k}}{s_i-s_1}\right] \tag{2}$$

is non-null. The integers $n_1, n_2, \ldots n_{k-1} \in \mathbb{N}$ are the number of times that runners 2, 3, ... k have crossed runner 1.

($\Leftarrow$). Assume the intersection is non-null. Then there exists a time T such that T is contained in the intersection. Each of the runners i, $2 \leq i \leq k$ are at least a distance $\frac{1}{k}$ from runner 1 at this time. This follows because each of the runners i have been running between times $\frac{n_{i-1}+\frac{1}{k}}{s_i-s_1}$ and $\frac{n_{i-1}+\frac{k-1}{k}}{s_i-s_1}$ and hence must be between a distance of $\frac{1}{k}$ and $\frac{k-1}{k}$ ahead of runner 1. Hence runner 1 becomes lonely at time T. □

Lemma 2. Let $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$. Then

$$\bigcap_{i=1}^{k}[a_i,b_i] \neq \emptyset. \tag{3}$$

if and only if $b_i \geq a_j$ for all $1 \leq i,j \leq k, j \neq i$.

Proof. ($\Rightarrow$). Suppose that

$$\bigcap_{i=1}^{k}[a_i,b_i] \neq \emptyset. \tag{4}$$

Suppose that $b_j < a_i$ for some $1 \leq i,j \leq k, j \neq i$. Then there must be some intervals $[a_j,b_j]$ and $[a_i,b_i]$ such that $[a_j,b_j] \cap [a_i,b_i] = \emptyset$. But this means that

$$\bigcap_{i=1}^{k}[a_i,b_i] = \emptyset \tag{5}$$

a contradiction.

($\Leftarrow$). We prove the contrapositive. Suppose that

$$\bigcap_{i=1}^{k}[a_i,b_i] = \emptyset. \tag{6}$$

Then there must exist some intervals $[a_m,b_m],[a_n,b_n]$ such that $[a_m,b_m] \cap [a_n,b_n] = \emptyset$ for $1 \leq n,m \leq k, n \neq m$. This implies that $b_m < a_n$ or $b_n < a_m$ and hence $b_i < a_j$ for some $1 \leq i,j \leq k, j \neq i$. Hence, if $b_i \geq a_j$ for all $1 \leq i,j \leq k, j \neq i$, then

$$\bigcap_{i=1}^{k}[a_i,b_i] \neq \emptyset. \tag{7}$$

□

Lemma 3. Let $s_1 < s_2 < \ldots < s_{i-1} < s_j < s_{i+1} < \ldots < s_k$ for $2 \leq i \leq k-1$. Then runner j becomes lonely if and only if there exists $q_1, q_2, \ldots q_{k-1} \in \mathbb{N}$ such that

$$\bigcap_{i=1}^{j-1} \left[\frac{q_i + \frac{1}{k}}{s_j - s_i}, \frac{q_i + \frac{k-1}{k}}{s_j - s_i} \right] \bigcap \bigcap_{i=j+1}^{k} \left[\frac{q_i + \frac{1}{k}}{s_i - s_j}, \frac{q_i + \frac{k-1}{k}}{s_i - s_j} \right] \neq \emptyset \tag{8}$$

Proof. ($\Rightarrow$). Suppose runner j becomes lonely. Then j becomes lonely from runners i for $1 \leq i \leq j-1$ and from runners i for $j+1 \leq i \leq k$. If j becomes lonely from i for $1 \leq i \leq j-1$, then since j is the fastest runner among these runners, j travels between a distance of $\frac{1}{k}$ and $\frac{k-1}{k}$ ahead of each of these runners. This required a time between $\frac{q_i + \frac{1}{k}}{s_j - s_i}$ and $\frac{q_i + \frac{k-1}{k}}{s_j - s_i}$ where $q_i \in \mathbb{N}$. Hence at some time T such that

$$T \in \bigcap_{i=1}^{j-1} \left[\frac{q_i + \frac{1}{k}}{s_j - s_i}, \frac{q_i + \frac{k-1}{k}}{s_j - s_i} \right] \tag{9}$$

runner j becomes lonely from each runner i with $1 \leq i \leq j-1$. Likewise, if j becomes lonely from runners i where $j+1 \leq i \leq k$, then since j is the slowest among these runners, it follows that each of the runners i have travelled a distance between $\frac{1}{k}$ and $\frac{k-1}{k}$ ahead of j. This required a time between $\frac{q_i + \frac{1}{k}}{s_i - s_j}$ and $\frac{q_i + \frac{k-1}{k}}{s_i - s_j}$. Hence, runner j becoming lonely from runners i implies that there exists some time T such that

$$T \in \bigcap_{i=j+1}^{k} \left[\frac{q_i + \frac{1}{k}}{s_i - s_j}, \frac{q_i + \frac{k-1}{k}}{s_i - s_j} \right] \tag{10}$$

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If runner j becomes lonely from runners i for $1 \leq i \leq j-1$ and from runners i for $j+1 \leq i \leq k$, then there must exist some time T such that

$$T \in \bigcap_{i=1}^{j-1} \left[\frac{q_i + \frac{1}{k}}{s_j - s_i}, \frac{q_i + \frac{k-1}{k}}{s_j - s_i} \right] \bigcap \bigcap_{i=j+1}^{k} \left[\frac{q_i + \frac{1}{k}}{s_i - s_j}, \frac{q_i + \frac{k-1}{k}}{s_i - s_j} \right]. \tag{11}$$

Hence the intersection must be non-null.

($\Leftarrow$). Suppose the intersection is non-null. Then there exists some time T such that

$$T \in \bigcap_{i=1}^{j-1} \left[\frac{q_i + \frac{1}{k}}{s_j - s_i}, \frac{q_i + \frac{k-1}{k}}{s_j - s_i} \right] \tag{12}$$

and

$$T \in \bigcap_{i=j+1}^{k} \left[\frac{q_i + \frac{1}{k}}{s_i - s_j}, \frac{q_i + \frac{k-1}{k}}{s_i - s_j} \right]. \tag{13}$$

Consider intersection (12). Since runner j is the fastest runner among the first i runners for $1 \leq i \leq j-1$, if there exists some time T in intersection (12), then there must be a time T such that runner j has travelled between a distance of $\frac{1}{k}$ and $\frac{k-1}{k}$ ahead of the first i runners. Hence, there must be a time T such that j becomes lonely from the first i runners. Likewise, if intersection (13) is non-null, then there exists some time T such that runner j becomes lonely from the fastest runners i where $j+1 \leq i \leq k$. This follows because at time T, each of the fastest runners i have travelled between a distance of $\frac{1}{k}$ and $\frac{k-1}{k}$ ahead of runner j. Hence if there exists some time T contained in both intersection (12) and (13), then runner j becomes lonely from both the slowest runners i for $1 \leq i \leq j-1$ and from the fastest runners i for $j+1 \leq i \leq k$ and thus becomes lonely. □

Lemma 4. Assume $s_1 < s_2 < \ldots < s_{k-1} < s_k$ with $k \geq 3$. Then runner k becomes lonely if and only if there exist $r_1, r_2, \ldots r_{k-1} \in \mathbb{N}$ such that

$$\bigcap_{i=1}^{k-1} \left[\frac{r_i + \frac{1}{k}}{s_k - s_i}, \frac{r_i + \frac{k-1}{k}}{s_k - s_i} \right] \neq \emptyset. \tag{14}$$

Proof. ($\Rightarrow$). Suppose runner k becomes lonely. Then runner k is at least a distance $\frac{1}{k}$ from runner i at some time T. This occurs when runner k has run between a distance of $\frac{1}{k}$ and $\frac{k-1}{k}$ ahead of runner i, requiring a time between $\frac{r_i + \frac{1}{k}}{s_k - s_i}$ and $\frac{r_i + \frac{k-1}{k}}{s_k - s_i}$. Hence, k becomes lonely from i during some time T $\in \left[\frac{r_i + \frac{1}{k}}{s_k - s_i}, \frac{r_i + \frac{k-1}{k}}{s_k - s_i} \right]$. Therefore, k becomes lonely from i for $1 \leq i \leq k-1$ if there exists some T such that

$$T \in \bigcap_{i=1}^{k-1} \left[\frac{r_i + \frac{1}{k}}{s_k - s_i}, \frac{r_i + \frac{k-1}{k}}{s_k - s_i} \right]. \tag{15}$$

This implies that runner k becomes lonely if

$$\bigcap_{i=1}^{k-1} \left[\frac{r_i + \frac{1}{k}}{s_k - s_i}, \frac{r_i + \frac{k-1}{k}}{s_k - s_i} \right] \neq \emptyset. \tag{16}$$

($\Leftarrow$). Suppose that

$$\bigcap_{i=1}^{k-1} \left[\frac{r_i + \frac{1}{k}}{s_k - s_i}, \frac{r_i + \frac{k-1}{k}}{s_k - s_i} \right] \neq \emptyset. \tag{17}$$

Then there exists some T in the intersection. At this time T, each runner i has been running for some time between $[\frac{r_i+\frac{1}{k}}{s_k-s_i}, \frac{r_i+\frac{k-1}{k}}{s_k-s_i}]$. This implies that runner k has travelled between a distance of $\frac{1}{k}$ and $\frac{k-1}{k}$ ahead of each runner i at time T. Hence, runner k becomes lonely from each runner i at time T. □

2 The Diophantine Approximation Problems

The LRC is equivalent to the following Diophantine approximation problems.

Theorem 1. Let $k \geq 5$. The LRC is equivalent to the following Diophantine approximation problems:

Slowest runner. Let $s_1 < s_2 < s_3 < \ldots < s_k$. Then there exists $n_1, n_2, \ldots n_{k-1} \in \mathbb{N}$ such that

$$\frac{kn_{m-1} + k - 1}{kn_{i-1} + 1} \geq \frac{s_m - s_1}{s_i - s_1} \geq \frac{kn_{m-1} + 1}{kn_{i-1} + k - 1} \tag{18}$$

for all $2 \leq i, m \leq k$, $m > i$.

Intermediate runner $j = 2$. There exists $m_{j2}, m_{21} \in \mathbb{N}$ such that

$$\frac{km_{j2} + k - 1}{km_{i2} + 1} \geq \frac{s_j - s_2}{s_i - s_2} \geq \frac{km_{j2} + 1}{km_{i2} + k - 1} \tag{19}$$

for $3 \leq j, i \leq k$ and $j > i$ and

$$\frac{km_{i2} + k - 1}{km_{21} + 1} \geq \frac{s_i - s_2}{s_2 - s_1} \geq \frac{km_{i2} + 1}{km_{21} + k - 1} \tag{20}$$

for $3 \leq i \leq k$.

Intermediate runner $3 \leq j \leq k-2$. There exists $q_{jm}, q_{ji}, q_{mj}, q_{ij}$ and $q'_{bj}, q'_{ja} \in \mathbb{N}$ such that

$$\frac{kq_{jm}+k-1}{kq_{ji}+1} \geq \frac{s_j-s_m}{s_j-s_i} \geq \frac{kq_{jm}+1}{kq_{ji}+k-1} \text{ for all } 1 \leq i, m \leq j-1, i < m, \quad (21)$$

$$\frac{kq_{mj}+k-1}{kq_{ij}+1} \geq \frac{s_m-s_j}{s_i-s_j} \geq \frac{kq_{mj}+1}{kq_{ij}+k-1} \text{ for all } j+1 \leq i, m \leq k-1, i < m, \quad (22)$$

$$\frac{kq'_{bj}+k-1}{kq'_{ja}+1} \geq \frac{s_b-s_j}{s_j-s_a} \geq \frac{kq'_{bj}+1}{kq'_{ja}+k-1} \text{ for all } 1 \leq a \leq j-1 \text{ and } j+1 \leq b \leq k \quad (23)$$

Intermediate runner $j = k-1$. There exists $m'_{k-1j}, m'_{kk-1} \in \mathbb{N}$ such that

$$\frac{km'_{k-1j}+k-1}{km'_{k-1i}+1} \geq \frac{s_{k-1}-s_j}{s_{k-1}-s_i} \geq \frac{km'_{k-1i}+1}{km'_{k-1i}+k-1} \quad (24)$$

for $3 \leq j, i \leq k$ and $j > i$ and

$$\frac{km_{kk-1}+k-1}{km_{k-1a}+1} \geq \frac{s_k-s_{k-1}}{s_{k-1}-s_a} \geq \frac{km_{kk-1}+1}{km_{k-1a}+k-1} \quad (25)$$

for $1 \leq a \leq k-2$.

Fastest runner. There exists $r_1, r_2, \ldots r_{k-1} \in \mathbb{N}$ such that

$$\frac{kr_m+k-1}{kr_i+1} \geq \frac{s_k-s_m}{s_k-s_i} \geq \frac{kr_m+1}{kr_i+k-1} \quad (26)$$

for all $1 \leq i, m \leq k-1, m > i$.

In the next section we prove this equivalency.

3 Proof of Equivalency

To prove that the LRC is equivalent to these Diophantine approximation problems, we will consider three cases; the case in which a runner has the slowest speed, the case in which the runner has an intermediate speed and the case in which a runner has the fastest speed. Since the cases $j = 2$ and $j = k-1$ are similar to the slowest and fastest cases, we will omit these cases for brevity's sake.

Theorem 1. The LRC is equivalent to the set of Diophantine approximation problems in section 2.

Proof. (LRC $\Rightarrow$ Diophantine). *Slowest runner.* Suppose the LRC is true for runners with speeds $s_1 < s_2 < s_3 < \ldots < s_k, k \geq 5$. Then the slowest runner must become lonely. By Lemma 1 this means that there exists $n_1, n_2, \ldots n_{k-1} \in \mathbb{N}$ such that

$$\bigcap_{i=2}^{k} \left[\frac{n_{i-1} + \frac{1}{k}}{s_i - s_1}, \frac{n_{i-1} + \frac{k-1}{k}}{s_i - s_1} \right] \neq \emptyset. \tag{27}$$

By Lemma 2 the above intersection is non-null if and only if

$$\frac{n_{i-1} + \frac{k-1}{k}}{s_i - s_1} \geq \frac{n_{m-1} + \frac{1}{k}}{s_m - s_1}. \tag{28}$$

for all $2 \leq i, m \leq k$, $m \neq i$. The above inequality implies that

$$\frac{s_m - s_1}{s_i - s_1} \geq \frac{kn_{m-1} + 1}{kn_{i-1} + k - 1} \tag{29}$$

for all $2 \leq i, m \leq k$, $m \neq i$. Hence

$$\frac{kn_{m-1} + k - 1}{kn_{i-1} + 1} \geq \frac{s_m - s_1}{s_i - s_1} \geq \frac{kn_{m-1} + 1}{kn_{i-1} + k - 1} \tag{30}$$

for all $2 \leq i, m \leq k$, $m > i$.

Intermediate runner $j = 2$. Similar to the slowest case.

Intermediate runner $3 \leq j \leq k - 2$.. The runner with intermediate speed must become lonely as well. Let the runner with intermediate speed have speed s_j where $s_1 < s_2 < \ldots < s_{i-1} < s_j < s_{i+1} < \ldots < s_k$ for $2 \leq i \leq k - 1$. Then by Lemma 3 this implies that there exists $q_{ji}, q_{ij} \in \mathbb{N}$ such that

$$\bigcap_{i=1}^{j-1} \left[\frac{q_{ji} + \frac{1}{k}}{s_j - s_i}, \frac{q_{ji} + \frac{k-1}{k}}{s_j - s_i} \right] \bigcap \bigcap_{i=j+1}^{k} \left[\frac{q_{ij} + \frac{1}{k}}{s_i - s_j}, \frac{q_{ij} + \frac{k-1}{k}}{s_i - s_j} \right] \neq \emptyset. \tag{31}$$

This implies that

$$\bigcap_{i=1}^{j-1} \left[\frac{q_{ji} + \frac{1}{k}}{s_j - s_i}, \frac{q_{ji} + \frac{k-1}{k}}{s_j - s_i} \right] \neq \emptyset, \tag{32}$$

$$\bigcap_{i=j+1}^{k} \left[\frac{q_{ij} + \frac{1}{k}}{s_i - s_j}, \frac{q_{ij} + \frac{k-1}{k}}{s_i - s_j} \right] \neq \emptyset. \tag{33}$$

By Lemma 2, the above intersections imply that

$$\frac{q_{jm} + \frac{k-1}{k}}{s_j - s_m} \geq \frac{q_{ji} + \frac{1}{k}}{s_j - s_i} \tag{34}$$

for $1 \leq i, m \leq j-1, m \neq i$ and

$$\frac{q_{mj} + \frac{k-1}{k}}{s_m - s_j} \geq \frac{q_{ij} + \frac{1}{k}}{s_i - s_j} \tag{35}$$

for $j+1 \leq i, m \leq k, m \neq i$. Hence it follows that

$$\frac{s_j - s_i}{s_j - s_m} \geq \frac{kq_{ji} + 1}{kq_{jm} + k - 1} \tag{36}$$

for $1 \leq i, m \leq j-1, m \neq i$ and

$$\frac{s_i - s_j}{s_m - s_j} \geq \frac{kq_{ij} + 1}{kq_{mj} + k - 1} \tag{37}$$

for $j+1 \leq i, m \leq k, m \neq i$. Since intersection (27) is non-null, there must exist some sets such that

$$\left[\frac{q_{ja} + \frac{1}{k}}{s_j - s_a}, \frac{q_{ja} + \frac{k-1}{k}}{s_j - s_a}\right] \bigcap \left[\frac{q_{bj} + \frac{1}{k}}{s_b - s_j}, \frac{q_{bj} + \frac{k-1}{k}}{s_b - s_j}\right] \neq \emptyset \tag{38}$$

for all $1 \leq a \leq j-1$ and $j+1 \leq b \leq k-1$. By Lemma 2, this implies that

$$\frac{s_j - s_a}{s_b - s_j} \geq \frac{kq_{ja} + 1}{kq_{bj} + k - 1} \text{ for all } 1 \leq a \leq j-1 \text{ and } j+1 \leq b \leq k-1, \tag{39}$$

$$\frac{s_b - s_j}{s_j - s_a} \geq \frac{kq_{bj} + 1}{kq_{ja} + k - 1} \text{ for all } 1 \leq a \leq j-1 \text{ and } j+1 \leq b \leq k-1. \tag{40}$$

Hence it follows that there exists $q_{ji}, q_{ij} \in \mathbb{N}$ such that

$$\frac{kq_{jm} + k - 1}{kq_{ji} + 1} \geq \frac{s_j - s_m}{s_j - s_i} \geq \frac{kq_{jm} + 1}{kq_{ji} + k - 1} \text{ for all } 1 \leq i, m \leq j-1, i < m, \tag{41}$$

$$\frac{kq_{mj} + k - 1}{kq_{ij} + 1} \geq \frac{s_m - s_j}{s_i - s_j} \geq \frac{kq_{mj} + 1}{kq_{ij} + k - 1} \text{ for all } j+1 \leq i, m \leq k-1, i < m, \tag{42}$$

$$\frac{kq'_{bj} + k - 1}{kq'_{ja} + 1} \geq \frac{s_b - s_j}{s_j - s_a} \geq \frac{kq'_{bj} + 1}{kq'_{ja} + k - 1} \text{ for all } 1 \leq a \leq j-1 \text{ and } j+1 \leq b \leq k \tag{43}$$

Intermediate runner $j = k-1$. Similar to the fastest runner covered next.

Fastest runner. The fastest runner must also become lonely. By Lemma 4, this implies that there exists $r_1, r_2, \ldots r_{k-1} \in \mathbb{N}$ such that

$$\bigcap_{i=1}^{k-1}\left[\frac{r_i+\frac{1}{k}}{s_k-s_i},\frac{r_i+\frac{k-1}{k}}{s_k-s_i}\right]\neq\emptyset. \tag{44}$$

By Lemma 2, this implies that

$$\frac{r_i+\frac{k-1}{k}}{s_k-s_i}\geq\frac{r_m+\frac{1}{k}}{s_k-s_m} \tag{45}$$

for $1\leq i,m\leq k, i\neq m$. Hence,

$$\frac{s_k-s_m}{s_k-s_i}\geq\frac{kr_m+1}{kr_i+k-1} \tag{46}$$

for $1\leq i,m\leq k, i\neq m$. Thus

$$\frac{kr_m+k-1}{kr_i+1}\geq\frac{s_k-s_m}{s_k-s_i}\geq\frac{kr_m+1}{kr_i+k-1} \tag{47}$$

for all $1\leq i,m\leq k-1, m>i$.

(Diophantine $\Rightarrow$ LRC). *Slowest runner.* Let $s_1<s_2<\ldots<s_k$ where $k\geq 5$. Since there exists $n_1,n_2,\ldots n_{k-1}\in\mathbb{N}$ such that

$$\frac{s_m-s_1}{s_i-s_1}\geq\frac{kn_{m-1}+1}{kn_{i-1}+k-1} \tag{48}$$

for $2\leq i,m\leq k$, $m\neq i$ it follows that there exists $n_1,n_2,\ldots n_{k-1}\in\mathbb{N}$ such that

$$\frac{n_{i-1}+\frac{k-1}{k}}{s_i-s_1}\geq\frac{n_{m-1}+\frac{1}{k}}{s_m-s_1} \tag{49}$$

for $2\leq i,m\leq k$, $m\neq i$. By Lemma 2, it follows that

$$\bigcap_{i=2}^{k}\left[\frac{n_{i-1}+\frac{1}{k}}{s_i-s_1},\frac{n_{i-1}+\frac{k-1}{k}}{s_i-s_1}\right]\neq\emptyset. \tag{50}$$

By Lemma 1, it follows that the slowest runner becomes lonely.

Intermediate runner. Let runner j have some intermediate speed where $s_1<s_2<\ldots<s_{i-1}<s_j<s_{i+1}<\ldots<s_k$ for $2\leq i\leq k-1$. Then there exists $q_1,q_2,\ldots q_{k-1}\in\mathbb{N}$ such that

$$\frac{s_j - s_m}{s_j - s_i} \geq \frac{kq_m + 1}{kq_i + k - 1} \text{ for } 1 \leq i, m \leq j-1, i \neq m, \tag{51}$$

$$\frac{s_m - s_j}{s_i - s_j} \geq \frac{kq_m + 1}{kq_i + k - 1} \text{ for } j+1 \leq i, m \leq k-1, i \neq m, \tag{52}$$

$$\frac{s_j - s_a}{s_b - s_j} \geq \frac{kq_a + 1}{kq_b + k - 1} \text{ for some } 1 \leq a \leq j-1 \text{ and } j+1 \leq b \leq k-1, \tag{53}$$

$$\frac{s_b - s_j}{s_j - s_a} \geq \frac{kq_b + 1}{kq_a + k - 1} \text{ for some } 1 \leq a \leq j-1 \text{ and } j+1 \leq b \leq k-1. \tag{54}$$

Consider inequality (43). This implies that

$$\frac{q_i + \frac{k-1}{k}}{s_j - s_i} \geq \frac{q_m + \frac{1}{k}}{s_j - s_m} \tag{55}$$

for $1 \leq i, m \leq j-1, i \neq m$. Hence, by Lemma 2, it follows that

$$\bigcap_{i=1}^{j-1} \left[\frac{q_i + \frac{1}{k}}{s_j - s_i}, \frac{q_i + \frac{k-1}{k}}{s_j - s_i} \right] \neq \emptyset \tag{56}$$

for $1 \leq i, m \leq j-1, i \neq m$. By Lemma 4, runner j becomes lonely from the slowest $j-1$ runners 1, 2, … $j-1$.

Fastest runner. Let $s_1 < s_2 < \ldots < s_k$ where $k \geq 3$. Then there exists $r_1, r_2, \ldots r_{k-1} \in \mathbb{N}$ such that

$$\frac{s_k - s_m}{s_k - s_i} \geq \frac{kr_m + 1}{kr_i + k - 1} \tag{57}$$

for all $1 \leq i, m \leq k-1, i \neq m$. Hence there exists $r_1, r_2, \ldots r_{k-1} \in \mathbb{N}$ such that

$$\frac{r_i + \frac{k-1}{k}}{s_k - s_i} \geq \frac{r_m + \frac{1}{k}}{s_k - s_m}. \tag{58}$$

By Lemma 2, it follows that

$$\bigcap_{i=1}^{k-1} \left[\frac{r_i + \frac{1}{k}}{s_k - s_i}, \frac{r_i + \frac{k-1}{k}}{s_k - s_i} \right] \neq \emptyset. \tag{59}$$

By Lemma 4, it follows that the fastest runner becomes lonely. Hence this set of Diophantine approximation problems is equivalent to the LRC. □

References

[1] Schmidt, Wolfgang M. *Diophantine Approximation.* Springer. 1980.