Relative Phase and Time States

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June 14, 2019

Abstract: Attempt to construct phase state with number state of a single oscillator has been failing superficially because numbers cannot be negative, but, fundamentally because there should be no absolute phase state and the single oscillator do not have relative phase. Relative phase state is constructed with number state of two oscillators, because number difference between the oscillators can be arbitrary large negative or positive. Similarly, relative time state is constructed with number state of two oscillators with different angular velocity.

In [1], attempt to construct phase state as $\sum_n \mathrm{e}^{in\theta}|n\rangle$ failed obviously because the formula involves just a single oscillator only to be able to represent, if any, absolute phase state, which should not exist. Though relative phase is briefly discussed in [1], its importance is overlooked.

At the bottom of page 58 in section V "Phase Difference of Two Oscillators" of [1], "state with phase difference θ" is defined using number state as:

$$|R,\theta\rangle = \sum_{m=0}^{R} \mathrm{e}^{im\theta}|m\rangle\,|R-m\rangle$$

where R is the total number of excitations and m is the number of excitations in the first oscillator.

By restricting R to be even and introducing N and n as $R=2N$ and $m=N+n$, we obtain:

$$|R,\theta\rangle = |2N,\theta\rangle = \mathrm{e}^{iN\theta}\sum_{n=-N}^{N} \mathrm{e}^{in\theta}|N+n\rangle\,|N-n\rangle$$

where absolute phase of $\mathrm{e}^{iN\theta}$ may be ignored. Then:

$$\lim_{N\to\infty}\langle\theta',2N|2N,\theta\rangle = \lim_{N\to\infty}\sum_{n'=-N}^{N}\sum_{n=-N}^{N}\langle N-n'|\langle N+n'|\mathrm{e}^{i(n\theta-n'\theta')}|N+n\rangle|N-n\rangle$$

$$= \lim_{N\to\infty}\sum_{n=-N}^{N}\mathrm{e}^{in(\theta-\theta')} = \sum_{n=-\infty}^{\infty}\delta(\theta-\theta'+2\pi n)$$

Thus, $\lim_{N\to\infty}\sum_{n=-N}^{N} \mathrm{e}^{in\theta}|N+n\rangle\,|N-n\rangle$ is the relative phase state. Note that, the state is, in a sense, classical, because the state involves infinite number of quanta, which is required by number phase uncertainty that number uncertainty of phase state must be infinite even though number cannot be negative

Similarly, using two oscillators with angular velocity difference between single excitation of $\Delta\omega > 0$, relative time state can be constructed starting from $\lim_{N\to\infty}\sum_{n=-N}^{N} \mathrm{e}^{in\Delta\omega t}|N+n\rangle|N-n\rangle$, though, result of inner product of such states is $\sum_{n=-\infty}^{\infty} \delta(\Delta\omega(t-t')+2\pi n)$. That is, to remove periodicity, we must make $\Delta\omega$ infinitely small. By making $\Delta\omega = \omega_0/N$, where ω_0 is the maximum angular velocity difference between oscillators, relative time state should be:

$$|t\rangle = \lim_{\omega_0\to\infty}\lim_{N\to\infty}\sum_{n=-N}^{N}\sqrt{\frac{\omega_0}{N}}\mathrm{e}^{in\omega_0 t/N}|N+n\rangle|N-n\rangle$$

as:

$$\langle t'|t\rangle = \lim_{\omega_0\to\infty}\lim_{N\to\infty}\sum_{n'=-N}^{N}\sum_{n=-N}^{N}\langle N-n'|\langle N+n'|\frac{\omega_0}{N}\mathrm{e}^{\frac{i\omega_0(nt-n't')}{N}}|N+n\rangle|N-n\rangle$$

$$= \lim_{\omega_0\to\infty}\lim_{N\to\infty}\sum_{n=-N}^{N}\mathrm{e}^{in\Delta\omega(t-t')}\Delta\omega = \lim_{\omega_0\to\infty}\int_{-\omega_0}^{\omega_0}\mathrm{e}^{i\omega(t-t')}d\omega$$

$$= \delta(t-t')$$

Not surprisingly, time state is obtained as the infinite energy limit of state, because energy time uncertainty needs time state has infinite energy uncertainty even though energy cannot be below that of ground state.

Just as absolute phase is physically meaningless, so is absolute time. Thus, it is essential that phase and time states are defined only as relative phase and time between two oscillators. Note that sin and cos operators in [1] implicitly assume some phase reference to distinguish sin and cos.

Reference

[1] L. Suskind, J. Glogower, "QUANTUM MECHANICAL PHASE AND TIME OPERATOR", *https://link.aps.org/pdf/10.1103/PhysicsPhysiqueFizika.1.49*, 1964.