Intimations of the Irrationality of π From Reflections of the Rational Root Test

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Abstract

The rational root test gives a means for determining if a root of a polynomial is rational. If none of the possible rational roots are roots, then if the roots are real, they must be irrational. Combining this observation with Taylor polynomials and the Taylor series for $\sin(x)$ gives intimations that π, and e, are likely irrational.

Introduction

One can quickly believe and recall the rational root test from the very easiest linear example possible. Consider

$$2x - 1.$$

The last or constant coefficient over the first or leading coefficient gives the only root $\frac{1}{2}$. Playing on a biblical refrain, the last shall be on top and the first on the bottom to give a possible rational root. The same test works for rational coefficients:

$$\frac{1}{2}x - 1$$

has root one over one half gives 2, correct.

What if there is no constant term. This can only mean $x = 0$ is a root and an x can be factored out. This is the case for Taylor polynomials for sin. For example, $x^9 - x^7 + x^5 - x^3 + x$ is such a factor-able Taylor polynomial for sin; we've dropped the coefficients. With the coefficients and factoring, we have

$$x\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!}\right) = x\left(\frac{x^8}{9!} - \frac{x^6}{7!} + \frac{x^4}{5!} - \frac{x^2}{3!} + 1\right). \quad (1)$$

We know $x = 0$ is a root of sin and all $n\pi$, n an integer are also roots. Although this finite polynomial, (1), is not equal to sin, the rational root test for this polynomial says a rational root will be a positive or negative factor of 9!, an integer.

In this paper we explore whether or not any high school algebra concepts coupled with Taylor series and polynomials for $\sin(x)$ give hints of the irrationality of π.

Intimation One

Here's a calculator trick for finding rational roots fast. We'll give a proof of concept. Consider

$$(3x - 5)(7x + 2) = 21x^2 - 29x - 10.$$

This would be pretty hard to factor using something like the AC-method. But, using the calculator's table feature with $\Delta = \frac{1}{21}$ does the trick. The table shows $-\frac{6}{21} = -\frac{2}{7}$ is a root and $\frac{5}{3} = \frac{35}{21}$ is too. This example shows the leading coefficient gives all the denominators possible for rational roots. One could write a program that accepts A, B, C coefficients, stores $Ax^2 + Bx + C$ in Y_1 and then using a for loop searches for $Y_1(NX) = 0$, with $X = \frac{1}{A}$. Try it.

What does it mean when the leading coefficient is $\frac{1}{n!}$ as it is in the Taylor polynomials, like (1)? Possible rational roots must be integers that divide $n!$. But π is not a positive integer. What is missing in this thought experiment is the consideration as to whether the roots of Taylor's polynomials approach the roots of the Taylor series? The end behavior of the polynomials can't be like sin, but other places must have the polynomials approach the sin curve and this implies the roots of sin are being approached by the polynomials.

Integers modulo a given natural number are periodic in the set of integers, so there is some clue that periodicity is possible. This leads us to intimation two.

Taylor polynomials converging to sin

Intimation Two

We know all $n\pi$ are roots of sin. Combining this with (1), we see the coefficients of a Taylor polynomial for sin can absorb the powers of n, yielding possible rational roots of the form

$$\frac{n!}{n^n} \text{ or } \frac{n!}{m^n},$$

where $m > n$. As there is no limit in the n in $n\pi$, the right hand fraction for any n degree polynomial gets as small as we please, smaller than any fraction we can imagine. So, if the leading coefficient is to small to accomodate any rational roots, no rational roots for the limit of Taylor polynomials, the Taylor series, are possible.

Intimation Three

Wallis's product is

$$\prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right) = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots = \frac{\pi}{2}.$$

So taking an even multiple of this product of fractions and sticking it into a Taylor polynomial, I mean series, is supposed to yield zero. Seems unlikely given the leading coefficient is $\frac{1}{n!}$. You do get for the limit coefficient something like an infinite numerator over an infinite denominator – sounds like an irrational to me.

What about e

e seems to be the mini-me for π. Minime It has Taylor polynomials and a Taylor series associated with it. Like π the leading coefficients are of the form $\frac{1}{n!}$. Unlike $\sin(n\pi)$, e^x is not a periodic function. But like the above suspicious reasoning, if one puts natural numbers nx into the e^x, the coefficients absorb the n and produce a arbitrary leading coefficient: $\frac{n!x}{n!} = x$. The constant is 1, so, using the rational root test, the only root is 1?

Horizontal transformations

We've used the idea of putting a multiple with the argument of a function a couple of times. In College Algebra this is termed a horizontal transformation. So, to be clear, $f(cx)$, if $c > 1$, compresses or shrinks the curve horizontally. One can understand this: $f(2x)$ will yield the same value as

$x = 1$ only now at $x = \frac{1}{2}$, sooner; if $c < 1$, the curve is stretched horizontally, like a slinky that maintains its height. One can understand this: $\frac{1}{2}x$ will take a larger number than the original $x = 1$ to deliver $f(1)$. It will take $x = 2$.

So: If a root is rational, say $f(\frac{p}{q}) = 0$ then there exists a natural number q such that $f(\frac{1}{q}x)$, a horizontal strecth, delivers a root at $x = p$, a natural number. If no such $\frac{1}{q}$ exists then any root must be irrational. Well what then can we say about any function with a series with a leading coefficient of the form

$$\frac{1}{n!}x^n = ((\frac{1}{n!})^{\frac{1}{n}}x)^n?$$

References

[1] P. Eymard and J.-P. Lafon, *The Number π*, American Mathematical Society, Providence, RI, 2004.