Mapping the Real Numbers to the Set of Natural Numbers

by Jim Rock

Abstract. The terminating decimal fractions in the open interval (0, 1) are put in one to one correspondence with the set of positive integers. This shows that attempting to map the set of real numbers to the natural numbers by listing them as infinite decimal fractions is futile. The real numbers in the closed interval [0, 1] are represented as the limit of partial decimal sums. We create a new version of set theory (Level Set Theory) in which the set of real numbers has the same cardinality as the set of natural numbers. It sets up a one to one mapping between the real numbers and the set of natural numbers.

The positive integers can be put in one to one correspondence with the terminating decimal fractions in the open interval (0, 1). $1 \rightarrow .1,\ 2 \rightarrow .2,\ \ldots,\ 10 \rightarrow .01,\ \ldots$ Each terminating decimal is the mirror image reflection through the decimal point of a positive integer. The mapping does not include any repeating decimal fractions. From this mapping the set of all rational numbers would appear to be uncountable.

This shows that attempting to map the real numbers in the closed interval [0, 1] to the natural numbers, by listing them as infinite decimal fractions is futile. Cantor's diagonal proof that the real numbers are an uncountable set can never even get started.

The problem is in defining the real numbers as the set of all infinite decimal expansions. That's vague. Most non-algebraic real numbers cannot be explicitly referenced. We need a new definition. Each real number S in the closed interval [0, 1] is:

$$\text{the limit } m\rightarrow\infty \quad n = 1 \text{ to } m,\ 0 \leq a_n \leq 9\ \sum a_n/10^n = S.$$
$$\text{For } n = 1 \text{ to } m,\ S - \sum a_n/10^n < 1/10^m.$$

We can make the difference between the partial sums and their limiting value S arbitrarily small.

A new non-hierarchical (Level Set Theory) is created when we let $1/\chi_0 = 0$.

Then the limit $x\rightarrow\infty\ f(x) = \chi_0 \leftrightarrow$ the limit $x\rightarrow\infty\ 1/f(x) = 0$.

The limit $n\rightarrow\infty\quad n = \chi_0$, and the limit $n\rightarrow\infty\ 2^n = 2^{\chi_0} = \chi_0$. Thus, $|2^{\chi_0}| = |\chi_0|$.

The limit $n\rightarrow\infty\ n^n = \chi_0^{\chi_0} = \chi_0$. Thus, $|\chi_0^{\chi_0}| = |\chi_0|$.

There is no hierarchy of infinites. The limit $n\rightarrow\infty\ 2^{2^n} = 2^{2^{\chi_0}} = \chi_0$. $|2^{2^{\chi_0}}| = |\chi_0|$.

Since the power set of the natural numbers has the same cardinality as the natural numbers, the set of all real numbers has the same cardinality as the set of natural numbers. We can put all the individual real numbers in a set and draw them out one by one: $S_0, S_1, S_2, \ldots$ without replacing. That sets up a one to one mapping of the real numbers to the set of natural numbers.

Explore the detailed proofs and fascinating consequences of $|2^{\chi_0}| = |\chi_0|$ in https://arxiv.org/abs/1002.4433 ***Addressing mathematical inconsistency: Cantor and Gödel refuted*** by J. A. Perez.