DISPROOF OF THE RIEMANN HYPOTHESIS

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Abstract. This paper disproves the Riemann hypothesis by generalizing the results from Titchmarsh's book The Theory of the Riemann Zeta-Function to rearrangements of conditionally convergent series that represent the reciprocal function of zeta. When one replaces the conditionally convergent series in Titchmarsh's theorems and consequent proofs by its rearrangements, the left hand sides of equations change, but the right hand sides remain invariant. This contradiction disproves the Riemann hypothesis.

1 Introduction

In his article [1], Riemann introduced his zeta function $\zeta(s)$, a complex function of a complex variable s. Zeta is analytic everywhere except at its only singularity, a simple pole at $s = 1$. It is known that trivial zeta roots are all simple and located at negative even integers. Zeta has infinitely many nontrivial roots, all of which are located on the critical strip $0 \leqslant Re(s) \leqslant 1$ at generally unknown locations.

The Riemann hypothesis is given by the following theorem.

Theorem 1. *All of the nontrivial roots of zeta are located on the critical line $Re(s) = 1/2$.*

The Riemann zeta function [2, p.1] is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \tag{1}$$

with the sum running over all natural numbers n. This series converges absolutely on the halfplane $Re(s) > 1$.

One can introduce the Möbius function $\mu(n)$, and Möbius invert the series (1). The Möbius inverse [2, p.3] of the series (1) defining zeta

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Date: July 16, 2019.

function is then

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \tag{2}$$

The series in Eq. (2) converges absolutely on the halfplane $1 < Re(s)$.

We abbreviate *the Riemann hypothesis* by *RH*.

2 The Perron formula variant

Let s be a complex variable and let $\sigma = Re(s)$ and $t = Im(s)$.

Introduce an arbitrary permutation $\psi(n) : \mathbb{N} \to \mathbb{N}$.

The following lemma is a variant of Titchmarsh's Lemma 3.12 [2, p.60].

Lemma 1. *Let, for any $\sigma > 1$,*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_{\psi(n)}}{\psi(n)^s} \tag{3}$$

where $a_n = O\left\{g(n)\right\}$ with $g(n)$ being non-decreasing, and

$$\sum_{n=1}^{\infty} \frac{\left|a_{\psi(n)}\right|}{\psi(n)^{\sigma}} = O\left\{\frac{1}{(\sigma-1)^a}\right\} \tag{4}$$

as $\sigma \to 1$. Then if $c > 0$, $\sigma + c > 1$, x is not an integer, and $N = \lfloor x \rfloor$,

$$\begin{aligned} \sum_{\substack{n\in\mathbb{N} \\ \psi(n)<x}} \frac{a_{\psi(n)}}{\psi(n)^s} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} f(s+w)\frac{x^w}{w} dw + \\ O\left\{\frac{x^c}{T(\sigma+c-1)^a}\right\} + O\left\{\frac{g(2x)x^{1-\sigma}\log x}{T}\right\} + \\ O\left\{\frac{g(N)x^{1-\sigma}}{T|x-N|}\right\} \end{aligned} \tag{5}$$

Proof. As in Titchmarsh's proof, the starting equation is

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\frac{x}{\psi(n)}\right)^w \frac{dw}{w} = \begin{cases} 1 + O\left\{\frac{(x/\psi(n))^c}{T\log\left(x/\psi(n)\right)}\right\}, \text{ if } \psi(n) < x \\ O\left\{\frac{(x/\psi(n))^c}{T\log\left(x/\psi(n)\right)}\right\}, \text{ if } \psi(n) > x \end{cases} \tag{6}$$

This is valid for any natural number $\psi(n)$.

Multiply by $a_{\psi(n)}\psi(n)^{-s}$ and sum over all n.

$$\sum_{\substack{n\in\mathbb{N}\\ \psi(n)<x}} \frac{a_{\psi(n)}}{\psi(n)^s} = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \sum_{n=1}^{\infty} \frac{a_n}{n^{s+w}} \frac{x^w}{w} dw + O\left\{ \frac{x^c}{T} \sum_{n=1}^{\infty} \frac{|a_n|}{n^{\sigma+c}\left|\log(x/n)\right|} \right\} \tag{7}$$

The right hand side here is identical to Titchmarsh's result [2, p.61]. Since the series in the error term converges absolutely, as clearly demonstrated in Titchmarsh's proof, we rearranged its terms appropriately without affecting the result. We rearranged the terms of the series in the integrand appropriately too, since $\sigma + c > 1$, and thus the series in the integrand converges absolutely to $f(s + w)$. One just estimates the error term the same way Titchmarsh did in his Lemma 3.12 now, and Lemma 1 follows. □

We next prove a variant of Titchmarsh's Theorem 14.25(A) [2, p.369].

Theorem 2. *Let $\psi(n)$ be a permutation of natural numbers n. On RH, the series*

$$\sum_{n=1}^{\infty} \frac{\mu(\psi(n))}{\psi(n)^s} \tag{8}$$

is convergent, and its sum is $1/\zeta(s)$ for every s such that $\sigma > 1/2$.

Proof. In Lemma 1, take $a_n = \mu(\psi(n))$, $f(s) = 1/\zeta(s)$, $c = 2$, and x half an odd integer. We obtain

$$\sum_{\substack{n\in\mathbb{N}\\ \psi(n)<x}} \frac{\mu(\psi(n))}{\psi(n)^s} = \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)} \frac{x^w}{w} dw + O\left(\frac{x^2}{T}\right) \tag{9}$$

The right hand side is exactly the same as in Titchmarsh's proof. Following Titchmarsh, in the limit $x \to \infty$ the right hand side tends to $1/\zeta(s)$, and the theorem follows. □

The next corollary follows from this theorem.

Corollary 1. *On RH, the series $\sum_n \mu(n)n^{-s}$ converges unconditionally on the halfplane $\sigma > 1/2$.*

Proof. On RH, $1/\zeta(s)$ is holomorphic and single valued on the halfplane $\sigma > 1/2$, and so all the rearrangements of $\sum_n \mu(n)n^{-s}$ are of the same magnitude by Theorem 2. $\square$

3 Disproof of RH

Theorem 3. *RH is not true.*

Proof. Corollary 1 does not hold, since $\sum_n \mu(n)n^{-s}$ does not converge absolutely, since [2, p.5, Eq. (1.2.7)] on the halfplane $\sigma > 1$ one finds $\sum_n |\mu(n)|n^{-r} = \zeta(r)/\zeta(2r)$, and hence $\sum_n |\mu(n)|n^{-r}$ diverges for $r \leqslant 1$. Hence, by the Riemann series theorem, $\sum_n \mu(n)n^{-s}$ does not converge unconditionally on the halfplane $\sigma > 1/2$. $\square$

4 Conclusions

This paper disproves the Riemann hypothesis by generalizing the results from Titchmarsh's book The Theory of the Riemann Zeta-Function to rearrangements of conditionally convergent series that represent the reciprocal function of zeta. When one replaces the conditionally convergent series in Titchmarsh's theorems and consequent proofs by its rearrangements, the left hand sides of equations change, but the right hand sides remain invariant. This contradiction disproves the Riemann hypothesis.

References

[1] B. Riemann, *On the Number of Prime Numbers less than a Given Quantity*, translated by David R. Wilkins, `https://www.claymath.org/sites/default/files/ezeta.pdf`

[2] E.C. Titchmarsh, *The theory of the Riemann zeta-function*, Oxford University Press, 1986, second edition