

# Proof of Twin Prime Conjecture

K.H.K. Geerasee Wijesuriya

Research Scientist in Physics and Astronomy

PhD student in Astrophysics, Belarusian National Technical University

BSc (Hons) in Physics and Mathematics University of Colombo, Sri Lanka

Doctorate Degree in Physics

geeraseew@gmail.com , geerasee1@gmail.com

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## **Author's Biography**

The author of this research paper is K.H.K. Geerasee Wijesuriya . And this proof of twin prime conjecture is completely K.H.K. Geerasee Wijesuriya's proof.

Geerasee is now 30 years old and she studied before at Faculty of Science, University of Colombo Sri Lanka. And she graduated with BSc (Hons) in Physics and Mathematics from the University of Colombo, Sri Lanka in 2014. And in March 2018, she completed her first Doctorate Degree in Physics with first class recognition. Now she is following her second PhD in Astrophysics with Belarusian National Technical University.

Geerasee has been invited by several Astronomy/Physics institutions and organizations worldwide, asking to get involve with them. Also, She has received several invitations from some private researchers around the world asking to contribute to their researches. She worked as Mathematics tutor/Instructor at Mathematics department, Faculty of Engineering, University of Moratuwa, Sri Lanka. Furthermore she has achieved several other scientific achievements already.

## **List of abbreviations**

Faculty of Science, University of Colombo, Sri Lanka , Belarusian National Technical University Belarus

## **Acknowledgement**

I would be thankful to my parents who gave me the strength to go forward with mathematics and Physics knowledge and achieve my scientific goals.

**Keywords:** prime; contradiction; greater than ; natural number

## **Abstract**

A twin prime numbers are two prime numbers which have the difference of 2 exactly. In other words, twin primes is a pair of prime that has a prime gap of two. Sometimes the term twin prime is used for a pair of twin primes; an alternative name for this is prime twin or prime pair. Up to date there is no any valid proof/disproof for twin prime conjecture. Through this research paper, my attempt is to provide a valid proof for twin prime conjecture.

## **Literature Review**

The question of whether there exist infinitely many twin primes has been one of the great open questions in number theory for many years. This is the content of the twin prime conjecture, which states that there are infinitely many primes  $p$  such that  $p + 2$  is also prime. In 1849, de Polignac made the more general conjecture that for every natural number  $k$ , there are infinitely many primes  $p$  such that  $p + 2k$  is also prime. The case  $k = 1$  of de Polignac's conjecture is the twin prime conjecture.

A stronger form of the twin prime conjecture, the Hardy–Littlewood conjecture, postulates a distribution law for twin primes akin to the prime number theorem. On April 17, 2013, Yitang Zhang announced a proof that for some integer  $N$  that is less than 70 million, there are infinitely many pairs of primes that differ by  $N$ . Zhang's paper was accepted by *Annals of Mathematics* in early May 2013. Terence Tao subsequently proposed a Polymath Project collaborative effort to optimize Zhang's bound. As of April 14, 2014, one year after Zhang's announcement, the bound has been reduced to 246. Further, assuming the Elliott–Halberstam conjecture and its generalized form, the Polymath project wiki states that the bound has been reduced to 12 and 6, respectively. These improved bounds were discovered using a different approach that was simpler than Zhang's and was discovered independently by James Maynard and Terence Tao.

## Assumption

Let's assume that there are finitely many twin prime numbers.

Therefore we proceed by considering that there are finitely many twin prime numbers. Then let the highest twin prime numbers are  $P_{n-1}$  and  $(P_{n-1} + 2)$ . Then for all prime numbers  $P_n$  greater than  $P_{n-1}$ ,  $(P_n - 2)$  is not a prime number.

## Methodology

With this mathematical proof, I use the contradiction method to prove the twin prime conjecture.

Let  $P_n$  is an arbitrary prime number greater than  $P_{n-1}$  (because there are infinite number of prime numbers). Then according to our consideration,  $(P_n - 2)$  is not a prime number. Since  $P_n > 2$  and since  $P_n$  is a prime number and since  $P_n$  is an odd number, for all prime numbers  $P_i$ :

$$P_i (< P_n / 2): P_n / P_i = r_1$$

$$\text{Thus } P_n = P_i * r_1 \dots \dots \dots (01)$$

Where  $r_1$  is a rational number (which is not a natural number)

But according to our consideration,  $(P_n - 2)$  is not a prime number. Also since  $P_n$  is a prime number greater than 2,  $(P_n - 2)$  is an odd number.

Thus for some prime number  $P_1 (< [(P_n - 2) / 2])$ ;  $(P_n - 2) / P_1 = x_1$ . Where we choose  $P_1$  such that  $x_1$  is a natural number. But since previously chose  $P_i$  is any arbitrary prime number less than  $(P_n / 2)$ ; now we consider  $P_1 = P_i$

$$\text{Then } (P_n - 2) = P_1 * x_1 \dots \dots \dots (02) \text{ and } P_n = P_1 * r_1 \dots \dots \dots (01)$$

Let  $P_N$  is a prime number (greater than  $P_n$ ). Then according to our assumption,  $(P_N + 2)$  is not a prime number. Here  $P_N$  is a prime number such that  $(P_N + 2)$  is dividing by prime number  $P_2$ .  
.....(1.1)

Thus  $(P_N + 2) = P_2 * x_2$  for some  $x_2$  natural number. Because there are infinitely many prime numbers.

Since  $P_N$  is a prime number, for some  $r_2$  (rational number which is not a natural number):  $P_N / r_2 = P_2$ .

Thus  $(P_N + 2) = P_2 * x_2$ .....(03) and  $P_N = r_2 * P_2$  .....(04)

$x_1$  and  $x_2$  are natural numbers and  $P_1$  and  $P_2$  are prime numbers.

Since  $P_N$  is a prime number,  $(P_N - 2)$  is also not a prime number ( Since  $P_N - 2 > P_{n-1}$  )

Then for some prime  $P_3$ ,  $(P_N - 2) / P_3 = x_3$

$(P_N - 2) = P_3 * x_3$  .....(05)

By (04) and (05):  $P_3 * x_3 = P_2 * r_2 - 2$  .....(06)

**But according to the below induction method proof which is in the "Proof" below, there exists primes  $P_n$  and  $P_N$  such that  $(P_N + 2)$  and  $(P_n - 2)$  both are divisible by 3 (where  $P_1 = 3$ ).**

**\*\*\* To see the induction method proof, please refer the 'Proof' below.**

**Then  $(P_N + 2) = (P_n - 2) + 3.l$  for some  $l$  even natural number.....(06)'**

**Then  $(P_N - 2) = (P_n - 2) + 3.l - 4 = P_n + 3.l - 6 = P_n + 3 * (l - 2)$ .**

**Since  $(P_N - 2)$  is divisible by  $P_3$ ,  $[ P_n + 3 * (l - 2) ]$  is divisible by  $P_3$ . .....(6.1)**

And we know that  $(P_N + 2) = (P_n - 2) + 3.l \rightarrow P_N = P_n + 3.l - 4$  .....(\*)

By (\*):  $P_1 * r_1 + 3.l - 4 = r_2 * P_2$ . Thus by (06):  $P_3 * x_3 + 2 = P_1 * r_1 + 3.l - 4$

Thus  $P_3 * x_3 - 3.l + 6 = P_1 * r_1$

$P_3 * x_3 - 3 * (l - 2) = P_1 * r_1$  .....(6.1.0)  $\rightarrow P_3 * x_3 - 3.l + 6 = P_1 * r_1 = P_n$ .

Thus  $P_3 * x_3 - 3 * (l - 2) = P_n \rightarrow P_3 * x_3 - 3 * (l - 2) + q = P_n + q = P_0$ . Where we choose  $q$  ( where  $6 \mid q$ ) an integer number such that  $P_n + q = P_0 =$  integer number that does not divide by  $P_3$ .

Then  $P_3 * x_3 - 3 \cdot (l - 2) + q = P_0 = P_3 \cdot r$

Then  $P_3 * x_3 - [3 \cdot (l - 2) - q] = P_0 = P_3 \cdot r \dots\dots\dots(6.1.0.1)$

Here  $r$  is some non-natural number. Because since  $P_0$  is an integer number that does not divide by  $P_3$ ,  $r$  is not an integer number, but  $r$  is a rational number.

**But by (6.1.0):**  $P_3 * x_3 - 3 \cdot (l - 2) = P_1 \cdot r_1 = P_n$ . Then  $[P_3 \cdot x_3 - P_n] = 3 \cdot (l - 2)$ ; since  $l$  is an even number,  $(3 \cdot l - 6)$  is divisible by 6. Thus  $[P_3 \cdot x_3 - P_n]$  is divisible by 6.....(6.1.1)

**But we choose**  $l = (q / 6) + 2 + \{ [P_3 \cdot x_3 - P_n] / 6 \}$  = a natural number (because by 6.1.1). Here we can adjust the value of  $[q / 6]$  (integer number) such that the output of

$[ (q / 6) + 2 + \{ [P_3 \cdot x_3 - P_n] / 6 \}]$  gives the value of  $l$  as in (06)'.

i.e. we can adjust the value of  $[q / 6]$  as 1, 2, 3, 4, ... or -1, -2, -3,... such that the output of

$[ (q / 6) + 2 + \{ [P_3 \cdot x_3 - P_n] / 6 \}]$  gives the value of  $l$  as in (06)'.

**Then:**  $6 \cdot l + P_n - 12 - q = P_3 \cdot x_3$ ; Where  $x_3$  is a natural number.

**Then**  $[P_n + 3 \cdot (l - 2)] + [3 \cdot (l - 2) - q] = P_3 \cdot x_3 \dots\dots\dots(6.2)$

**By (6.1) we know that**  $(P_n + 3 \cdot (l - 2))$  is divisible by  $P_3$ . Since  $x_3$  is a natural number, by (6.2):  $(3 \cdot l - 6 - q)$  is divisible by  $P_3 \dots\dots\dots(6.3)$

Thus we know that  $[3 \cdot l - 6 - q]$  is divisible by  $P_3$  (by (6.3))

Thus by (6.1.0.1):  $P_3 * [x_3 - l_0] = P_3 \cdot r$ ; where  $l_0 = (3 \cdot l - 6 - q) / P_3 =$  integer number (by (6.3)).

Thus,  $x_3 - l_0 = r \dots\dots\dots(07)$  where  $l_0 = [3 \cdot l - 6 - q] / P_3 =$  an integer number.

But  $[x_3 - l_0]$  is an integer number. But  $r$  is not an integer number.

Thus by (07), there is a contradiction.

Therefore the only possibility is: our assumption is false.

Therefore there are infinitely many Twin Prime Numbers.

## Proof

Now let's prove that there exists infinite number of Prime numbers  $P_n$  such that  $3 \mid (P_n - 2)$ , by using mathematical induction method as below.

Let's consider the statement  $Q(n) : [P(n) - 2] / 3 = x(n)$ ; where  $P(n)$  is the  $n$ th prime number which obeys  $P(n) - 2 = 3 \cdot x(n)$ . And the meaning of  $x(n)$  is similar to that.

$Q(1)$ :  $[5 - 2] / 3 = 1 = x(1) =$  a natural number. Thus for  $n = 1$ , the result holds.

Now assume for  $n = s$ , the result  $Q(s)$  holds. Then  $[P_s - 2] / 3 = x(s) =$  natural number.

Here we must consider  $n = s$  part as below.

Let  $\epsilon_s$  is a positive real number  $\epsilon_s = [ - B + P_s + C_s - 2 + 3 \cdot k' ] / P_s > 0$  for all  $s > (M - 2)$ ,  $h_s < P_s * \epsilon_s$  (since the only existing  $s > (M - 2)$  is  $(M - 1)$ ; " for all  $s > (M - 2)$  means  $s = (M - 1)$  ). Where  $k'$  is an integer number. Here the chosen  $k'$  integer number is responsible for  $h_s < P_s * \epsilon_s$  for all  $s > (M - 2)$  and  $k'$  is responsible for  $\epsilon_{M-1} > 0$ . That means here the value of  $k'$  is responsible to say : " $\epsilon_s$  is existing such that  $h_s < P_s * \epsilon_s$ , only for  $s = (M-1)$ ". Here  $h_j = b_j$  for all  $j < (M - 1) = s$ . And where  $\sum b_j = B$  for  $j < (M - 1) = s$ . Then for some  $C_s$ ,  $h_s = P_s * \epsilon_s - C_s$ ; here  $s \equiv M - 1$ . \*\*\* the meaning of 'j' is the order number and  $h_j$  is the prime gap between  $P_{j+1}$  and  $P_j$ , please refer the below content and the 2<sup>nd</sup> reference.

But  $s \equiv (M - 1)$ . But here we chose  $C_{M-1}$  such that  $h_{M-1} = P_{M-1} * \epsilon_{M-1} - C_{M-1}$

But  $h_{M-1} = P_{M-1} * \epsilon_{M-1} - C_{M-1} = (P_s - B - 2 + 3 \cdot k')$ . Where  $k'$  is an integer number.

Then let's show for  $n = s + 1$ ,  $Q(s+1)$  holds. We denote  $P(s+1) = P_M$

But we know  $[P_s - 2] / 3 = x(s) \dots\dots\dots(8.1)$

Now let's use the 2<sup>nd</sup> reference to proceed further.

By 2<sup>nd</sup> reference,  $P_M = 2 + \sum_{j=1}^{M-1} h_j \dots\dots\dots(i)$

But we know already that for  $\epsilon_{M-1} > 0$ ,  $h_{M-1} < P_{M-1} * \epsilon_{M-1}$  for  $M - 1 = s$ .

Here  $s \equiv (M - 1)$

(\*\*\* refer the 2<sup>nd</sup> reference below)

Then we already know that for some  $C_{M-1}$  positive number,  $h_{M-1} = P_{M-1} * C_{M-1} - C_{M-1}$ .

But  $h_{M-1} = P_{M-1} * C_{M-1} - C_{M-1}$  for  $(M - 1) \equiv s$

We know already that  $C_{M-1} = [P_s - B + C_{M-1} - 2 + 3.k'] / P_{M-1} > 0$ .

And  $h_{M-1} = P_{M-1} * C_{M-1} - C_{M-1} = (-B + P_s - 2 + 3.k')$ . Where  $k'$  is an integer number. We know already that the chosen  $k'$  integer number is responsible for  $C_{M-1} > 0$ .

We know that  $h_j = b_j$  for all  $j < (M - 1)$ . Where  $b_j$  is a natural number. Also we know that  $\sum b_j = B$  for  $j < M - 1$ .

Thus by (i):  $P_M = 2 + P_s + 3.k' - B - 2 + B = 3.k' + P_s$

Thus  $(P_M - 2) = (P_s - 2) + 3.k' \dots\dots\dots(8.2)$

But  $(P_s - 2)$  is divisible by 3 ( $= P_1$ ) according to (8.1). Thus  $(P_M - 2)$  is divisible by 3 ( $= P_1$ ) according to (8.2), since  $3.k'$  is divisible by 3.

Thus  $(P_M - 2)$  is divisible by 3 ( $= P_1$ ). i.e.  $[P(s+1) - 2]$  is divisible by 3 ( $= P_1$ ).

Thus for  $n = s + 1$ , the result  $Q(n + 1)$  holds. Thus by mathematical induction method:

There exists infinite number of prime numbers  $P_M$  such that  $3 | (P_M - 2)$ .

Thus there exists  $P_n$  prime (where we consider them as prime numbers greater than  $P_{n-1}$ ) such that  $(P_n - 2)$  is divisible by 3 ( $= P_1$ ). Thus there exists  $P_n$  prime (greater than  $P_{n-1}$ ) such that  $(P_n - 2)$  is divisible by  $P_1 (=3)$ .

Now let's prove that there exists infinite number of Prime numbers  $P_N$  such that  $3 | (P_N + 2)$ , by using mathematical induction method as below.

Let's consider the statement  $Q(n) : [P(n) + 2] / 3 = x(n)$ ; where  $P(n)$  is the  $n$ th prime number which obeys  $P(n) + 2 = 3 \cdot x(n)$ . And the meaning of  $x(n)$  is similar to that.

$Q(1): [7 + 2] / 3 = 3 = x(1) =$  a natural number. Thus for  $n = 1$ , the result holds.



Now assume for  $n = s$ , the result  $Q(s)$  holds. Then  $[P_s + 2] / 3 = x(s) = \text{natural number}$ .

Here we must considered  $n = s$  part as below.

Let  $C_s$  is a positive real number  $C_s = [ - A + P_s + C_s - 2 + 3.k'' ] / P_s > 0$ , such that  $g_s < P_s * C_s$  for all  $s > (L - 2)$ . (Here  $s$  is going from 1 to  $(L - 1)$ ). Then "for all  $s > (L - 2)$ " means  $s = (L - 1)$ . Where  $k''$  is an even integer number. Here the chosen  $k''$  integer number is responsible for  $g_s < P_s * C_s$  for all  $s > (L - 2)$  (i.e.  $s = (L - 1)$ ) and  $C_{L-1} > 0$ . That means here the value of  $k''$  is responsible to say " $C_s$  is existing such that  $g_s < P_s * C_s$ , only for  $s = (L-1)$ ". Here  $g_j = a_j$  for all  $j < (L - 1) = s$ . And where  $\sum a_j = A$  for  $j < (L - 1) = s$ . Then for some  $C_s$ ,  $g_s = P_s * C_s - C_s$ ; here  $s \equiv L - 1$ . \*\*\* the meaning of 'j' is the order number and  $g_i$  is the prime gap between  $P_{j+1}$  and  $P_j$ . Please refer the below content and the 2<sup>nd</sup> reference.

But  $s \equiv (L - 1)$

But here we chose  $C_{L-1}$  such that  $g_{L-1} = P_{L-1} * C_{L-1} - C_{L-1}$

But  $g_{L-1} = P_{L-1} * C_{L-1} - C_{L-1} = (P_s - A - 2 + 3.k'')$ . Where  $k''$  is an integer number.

Then let's show for  $n = s + 1$ ,  $Q(s+1)$  holds. We denote  $P(s+1) = P_L$

But we know  $[P_s + 2] / 3 = x(s) \dots\dots\dots(9.1)$

Now let's use the 2<sup>nd</sup> reference to proceed further.

By 2<sup>nd</sup> reference,  $P_L = 2 + \sum_{j=1}^{L-1} g_j \dots\dots\dots(ii)$

But we know already that for  $C_{L-1} > 0$ ,  $g_{L-1} < P_{L-1} * C_{L-1}$ . Here  $s \equiv (L - 1)$

(\*\*\* refer the 2<sup>nd</sup> reference below)

Then we already know that for some  $C_{L-1}$  positive number,  $g_{L-1} = P_{L-1} * C_{L-1} - C_{L-1}$ .

But  $g_{L-1} = P_{L-1} * C_{L-1} - C_{L-1}$  for  $(L - 1) \equiv s$

We know already that  $C_{L-1} = [P_s - A + C_{L-1} - 2 + 3.k''] / P_{L-1} > 0$ .

And  $g_{L-1} = P_{L-1} * C_{L-1} - C_{L-1} = (- A + P_s - 2 + 3.k'')$ . Where  $k''$  is an integer number. We know already that the chosen  $k''$  integer number is responsible for  $C_{L-1} > 0$ .

We know that  $g_j = a_j$  for all  $j < (L-1)$ . Where  $a_j$  is a natural number. Also we know that  $\sum a_i = A$  for  $j < L-1$ .

Thus by (ii):  $P_L = 2 + P_s + 3.k'' - A - 2 + A = 3.k'' + P_s$

Thus  $(P_L + 2) = (P_s + 2) + 3.k'' \dots\dots\dots(9.2)$

But  $(P_s + 2)$  is divisible by 3 ( $= P_1$ ) according to (9.1). Thus  $(P_L + 2)$  is divisible by 3 ( $=P_1$ ) according to (9.2), since  $3.k''$  is divisible by 3.

Thus  $(P_L + 2)$  is divisible by 3 ( $= P_1$ ). i.e.  $[P(s+1) + 2]$  is divisible by 3 ( $= P_1$ ).

Thus for  $n = s + 1$ , the result  $Q(n + 1)$  holds. Thus by mathematical induction method:

There exists infinite number of prime numbers  $P_L$  such that  $3 | (P_L + 2)$ .

Thus there exists  $P_N$  prime (where we consider them as prime numbers greater than  $P_{n-1}$ ) such that  $(P_N + 2)$  is divisible by 3 ( $=P_1$ ). Thus  $(P_N + 2)$  is divisible by  $P_1 (=3)$ .

Therefore there exists infinite number of primes  $P_n$  and  $P_N$  such that  $3 | (P_n - 2)$  and  $3 | (P_N + 2)$ .

## Discussion

We assumed initially that there are finitely many twin primes. After proceeding with that, I ended up with a contradiction. But to get the contradiction, I used that  $P_n$  and  $P_N$  as primes numbers greater than  $P_{n-1}$ . Also to get the contradiction, I used the facts that  $(P_n - 2)$  and  $(P_N + 2)$  and  $(P_N - 2)$  as non-primes. And also I have used that  $x_1$ ,  $x_2$  and  $x_3$  as natural numbers (since  $(P_n - 2)$ ,  $(P_N + 2)$  and  $(P_N - 2)$  are not prime numbers). Therefore to get the contradiction, I have used the facts got from our assumption. Then the only possibility is our assumption is false.

## Results

Therefore I have used our assumption to get a contradiction finally as showed in (07). Therefore it is possible to conclude that our assumption is false.

**Thus there are infinitely many twin prime numbers.**

## Appendix

Prime number: A natural number which divides by 1 and itself only.

Twin Prime Numbers: Two prime numbers which have the difference exactly 2.

We denote 'i' th prime gap  $g_i = P_{i+1} - P_i$

Then according to the 2<sup>nd</sup> reference; Prime number  $P_N = 2 + \sum_{j=1}^{N-1} g_j$

Also by 2<sup>nd</sup> reference: for all  $\epsilon > 0$ , there is a natural number 'n' such that for all  $N - 1 > n$ ;

$$g_{N-1} < P_{N-1} \cdot \epsilon$$

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