Relation between mean proportionals of the parts and the whole of a line segment

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Abstract
Galileo derived a result for the relation between the two mean proportionals of the parts and the whole of a given line segment. He derived it for the internal division of the line segment. We derive in this note, a corresponding result for the external division of a given line segment.

Keywords: Mean proportional, internal division, external division, Semicircle, Right triangle

Introduction

Galileo used proportionals and mean proportionals extensively in his geometrical analysis of motion. An interesting and useful result there, concerns the relation between the mean proportionals of the lengths of the parts and the length of a given line segment, to the length of the entire line segment[1]. The relation being that: If a line segment is divided into two parts and the mean proportionals of each part and the whole line segment are taken, then the sum of the squares of these mean proportionals is equal to the square of the whole line segment. This result was demonstrated by Galileo for the *internal division* of a line segment. It would be interesting to also have the proof of the result for the *external division* of the line segment. I was curious to see if it could also be easily obtained by a similar geometrical method. I found it is possible. I present the result in this article.

Division of a line segment into two parts in two ways

There are two possible methods of dividing a given line segment AB into two parts in a given ratio. One is an internal division and the other is an external division. Let us consider both cases for our purpose at hand. Let AB be the given line segment (see Fig. 1). Let us cut it into two portions AC, CB at C, any point whatever, on AB. Two possible cases arise: 1) C lies in between A and B and 2) C lies on AB extended. The former division is called internal division and the later division the external division.

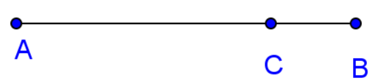

Fig. 1a Line segment AB is divided internally by point C.

Fig. 1b Line segment AB is divided externally by point C'.

In Figs 1a. 1b, C and C′ divide AB in the same ratio internally and externally. That is, C and C′ are harmonic conjugates.

Mean proportionals

Let AD represent the mean proportional between the whole line segment AB and the portion AC. Similarly, let BD represent the mean proportional between the whole line segment AB and the portion BC. We can show these relations as:

$$AB : AD = AD : AC \qquad (1)$$

$$AB : BD = BD : BC \qquad (2)$$

$$AB \cdot AC = AD^2 \qquad (3)$$

$$AB \cdot BC = BD^2 \qquad (4)$$

Galileo shows us a simple geometric method of obtaining these mean proportionals AD and BD.

Statement of the problem[1]

"If a straight line be cut at any point whatever and mean proportionals between this line and each of its parts be taken, the sum of the squares of these mean proportionals is equal to the square of the entire line."

Proof: a) Internal division

Let AB represent the given line segment. Draw a semi circle with AB as diameter (see Fig. 2). Choose any point C between A and B. Draw a perpendicular to AB at C to intersect the semicircle at D. Join A, D; B, D; and C, D. AD gives the mean proportional between the whole line segment AB and the portion AC. Similarly, BD gives the mean proportional between the whole line segment AB and BC.

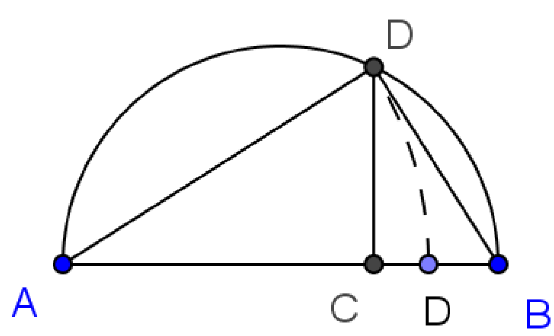

Fig. 2 Relation between a line segment and the mean proportinal with its parts. Case of internal division

What Galileo proved is that the sum of the squares of the mean proportional AD and BD is equal to the square of AB the whole line segment. Since ADB is a triangle in a semi circle it is a right triangle with right angle at D. Therefore, we easily see from Pythagoras theorem that

$$AD^2 + BD^2 = AB^2 \qquad (5)$$

AD and BD being the mean proportionals, we find the statement proved.

The above proof applies when C divides AB internally.

b) External division

We now prove the statement when C divides AB externally, in the same ratio as it was for the internal division.

We should note that the phrase ‘sum of the squares’ now becomes the difference of the squares. That means, we take the ‘sum’ as algebraic sum, so that it applies to both cases.

Let the point C′ be the point on AB extended, which divides AB externally in the same ratio as C divides AB internally. In other words, let C and C′ be harmonic conjugates. Draw a semi circle with AC′ as diameter (see Fig. 3). Draw a perpendicular to AB at B to intersect the semicircle at D′. Join A, D′; B, D′. AD′ gives the mean proportional between the whole line segment AB and the portion AC′.

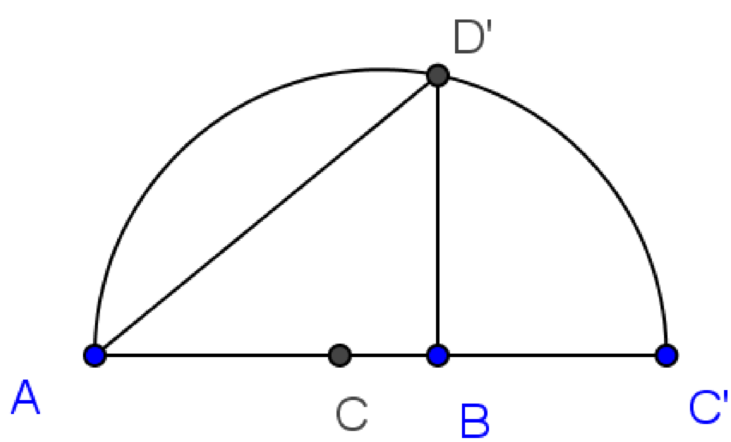

Fig. 3 Relation between a line segment and the mean proportinal with its parts. Case of external division

Similarly, BD′ gives the mean proportional between the whole line segment AB and BC′. Evidently, ABD′ is a right triangle. Using Pythagoras theorem again, we get,

$$(AD')^2 - (D'B)^2 = (AB)^2 \quad (6)$$

Thus we prove that the algebraic sum of the squares of the mean proportionals of the parts and the whole of a given line segment is equal to the square of the given line segment. This applies to both cases whether the given line segment is divided internally or externally.

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