A Relational Formulation of Classical Mechanics

Alfonso A. Blato

(2019) Buenos Aires
Argentina

This paper presents a relational formulation of classical mechanics which is invariant under transformations between inertial and non-inertial reference frames and which can be applied in any reference frame without introducing fictitious forces.

Introduction

The relational formulation of classical mechanics presented in this paper is obtained starting from a new force of interaction (called kinetic force, since this new force of interaction is directly related to kinetic energy)

The kinetic force $\mathbf{K}_{ij}$ exerted on a particle i of mass m_i by another particle j of mass m_j, caused by the interaction between particle i and particle j, is given by:

$$\mathbf{K}_{ij} \;=\; -\,\frac{m_i\,m_j}{M}\,\Big[\,(\vec{a}_i-\vec{a}_j)-2\,\vec{\omega}\times(\vec{v}_i-\vec{v}_j)+\vec{\omega}\times[\,\vec{\omega}\times(\vec{r}_i-\vec{r}_j)\,]-\vec{\alpha}\times(\vec{r}_i-\vec{r}_j)\,\Big]$$

where $\vec{a}_i$, $\vec{v}_i$, $\vec{r}_i$ are the acceleration, the velocity and the position of particle i, $\vec{a}_j$, $\vec{v}_j$, $\vec{r}_j$ are the acceleration, the velocity and the position of particle j, and M, $\vec{\omega}$, $\vec{\alpha}$ are the mass, the angular velocity and the angular acceleration of the Universe (see Annex I)

From the above equation it follows that the net kinetic force $\mathbf{K}_i$ acting on a particle i of mass m_i is given by:

$$\mathbf{K}_i \;=\; -\,m_i\,\Big[\,(\vec{a}_i-\vec{A})-2\,\vec{\omega}\times(\vec{v}_i-\vec{V})+\vec{\omega}\times[\,\vec{\omega}\times(\vec{r}_i-\vec{R})\,]-\vec{\alpha}\times(\vec{r}_i-\vec{R})\,\Big] \quad [\,\text{Eq. 2}\,]$$

where $\vec{R}$, $\vec{V}$ and $\vec{A}$ are the position, the velocity and the acceleration of the center of mass of the Universe.

The magnitudes $[\,m_i,\, m_j,\, M,\, \mathbf{K}_{ij},\, \mathbf{K}_i\,]$ are invariant under transformations between inertial and non-inertial reference frames.

Any reference frame S is an inertial reference frame when the angular velocity $\vec{\omega}$ of the Universe and the acceleration $\vec{A}$ of the center of mass of the Universe are equal to zero ($\vec{\omega}=0$ and $\vec{A}=0$) relative to S. Therefore, the reference frame S is a non-inertial reference frame when the angular velocity $\vec{\omega}$ of the Universe or the acceleration $\vec{A}$ of the center of mass of the Universe are not equal to zero ($\vec{\omega}\neq 0$ or $\vec{A}\neq 0$) relative to S.

Equation of Motion

The total force $\mathbf{T}_i$ acting on a particle i is always in balance.

$$\mathbf{T}_i \;=\; 0$$

If the total force $\mathbf{T}_i$ is divided into the following two parts: the net kinetic force $\mathbf{K}_i$ and the net dynamic force $\mathbf{F}_i$ ($\sum$ of gravitational forces, electrostatic forces, etc.) then we have:

$$\mathbf{K}_i + \mathbf{F}_i \;=\; 0$$

Now, substituting $\mathbf{K}_i$ by $\big[\,\mathrm{Eq.}\,2\,\big]$ dividing by m_i and rearranging, we obtain:

$$\vec{a}_i \;=\; \mathbf{F}_i/m_i + \vec{A} + 2\,\vec{\omega}\times(\vec{v}_i - \vec{V}) - \vec{\omega}\times[\,\vec{\omega}\times(\vec{r}_i - \vec{R})\,] + \vec{\alpha}\times(\vec{r}_i - \vec{R})$$

From the above equation it follows that particle i can have a non-zero acceleration even if there is no dynamic force acting on particle i, and also that particle i can have zero acceleration (state of rest or of uniform linear motion) even if there is an unbalanced net dynamic force acting on particle i.

However, from the above equation it also follows that Newton's first and second laws are valid in any inertial reference frame, since the angular velocity $\vec{\omega}$ of the Universe and the acceleration $\vec{A}$ of the center of mass of the Universe are equal to zero relative to any inertial reference frame.

General Observations

All equations of this paper can be applied in any inertial reference frame and also in any non-inertial reference frame.

Additionally, inertial and non-inertial observers must not introduce fictitious forces into $\mathbf{F}_i$.

In this paper, the following magnitudes $[\,m,\,\mathbf{r},\,\mathbf{v},\,\mathbf{a},\,M,\,K,\,\mathbf{T},\,\mathbf{K},\,\mathbf{F}\,]$ are invariant under transformations between inertial and non-inertial reference frames.

The kinetic forces are caused by the interactions between the particles and the net kinetic force is the force that balances the net dynamic force in each particle of the Universe.

In addition, the kinetic forces obey Newton's third law in its weak form or in its strong form and remain invariant under transformations between inertial and non-inertial reference frames (as all dynamic forces do)

On the other hand, this paper does not contradict Newton's first and second laws since these two laws are valid in any inertial reference frame (in Newtonian mechanics the kinetic forces are completely excluded)

Finally, the relational mechanics presented in this paper is observationally equivalent to Newtonian mechanics. However, non-inertial observers can only use Newtonian mechanics if they introduce fictitious forces into $\mathbf{F}_i$.

Annexes

Relational Universe

In classical mechanics, the Universe is a system that contains all particles, that is always free of external forces, and that all internal dynamic forces always obey Newton's third law in its weak form and in its strong form.

The position $\vec{R}$, the velocity $\vec{V}$ and the acceleration $\vec{A}$ of the center of mass of the Universe relative to a reference frame S (and the angular velocity $\vec{\omega}$ and the angular acceleration $\vec{\alpha}$ of the Universe relative to the reference frame S) are given by:

$$M \doteq \sum_i^{All} m_i$$

$$\vec{R} \doteq M^{-1} \sum_i^{All} m_i \, \vec{r}_i$$

$$\vec{V} \doteq M^{-1} \sum_i^{All} m_i \, \vec{v}_i$$

$$\vec{A} \doteq M^{-1} \sum_i^{All} m_i \, \vec{a}_i$$

$$\vec{\omega} \doteq \overleftrightarrow{I}^{-1} \cdot \vec{L}$$

$$\vec{\alpha} \doteq d(\vec{\omega})/dt$$

$$\overleftrightarrow{I} \doteq \sum_i^{All} m_i \, [\, |\, \vec{r}_i - \vec{R}\,|^2 \, \overleftrightarrow{1} - (\vec{r}_i - \vec{R}) \otimes (\vec{r}_i - \vec{R})\,]$$

$$\vec{L} \doteq \sum_i^{All} m_i \, (\vec{r}_i - \vec{R}) \times (\vec{v}_i - \vec{V})$$

where M is the mass of the Universe, $\overleftrightarrow{I}$ is the inertia tensor of the Universe (relative to $\vec{R}$) and $\vec{L}$ is the angular momentum of the Universe relative to the reference frame S.

Invariant Magnitudes

$$(\vec{r}_i - \vec{R}) \doteq \mathbf{r}_i = \mathbf{r}'_i$$

$$(\vec{r}'_i - \vec{R}') \doteq \mathbf{r}'_i = \mathbf{r}_i$$

$$(\vec{v}_i - \vec{V}) - \vec{\omega} \times (\vec{r}_i - \vec{R}) \doteq \mathbf{v}_i = \mathbf{v}'_i$$

$$(\vec{v}'_i - \vec{V}') - \vec{\omega}' \times (\vec{r}'_i - \vec{R}') \doteq \mathbf{v}'_i = \mathbf{v}_i$$

$$(\vec{a}_i - \vec{A}) - 2\, \vec{\omega} \times (\vec{v}_i - \vec{V}) + \vec{\omega} \times [\, \vec{\omega} \times (\vec{r}_i - \vec{R})\,] - \vec{\alpha} \times (\vec{r}_i - \vec{R}) \doteq \mathbf{a}_i = \mathbf{a}'_i$$

$$(\vec{a}'_i - \vec{A}') - 2\, \vec{\omega}' \times (\vec{v}'_i - \vec{V}') + \vec{\omega}' \times [\, \vec{\omega}' \times (\vec{r}'_i - \vec{R}')\,] - \vec{\alpha}' \times (\vec{r}'_i - \vec{R}') \doteq \mathbf{a}'_i = \mathbf{a}_i$$

Appendix A

Fields and Potentials I

The net kinetic force $\mathbf{K}_i$ acting on a particle i of mass m_i can also be expressed as follows:

$$\mathbf{K}_i \;=\; +\,m_i \Big[\, \mathbf{E} + (\vec{v}_i - \vec{V}\,) \times \mathbf{B} \,\Big]$$

$$\mathbf{K}_i \;=\; +\,m_i \Big[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + (\vec{v}_i - \vec{V}\,) \times (\nabla \times \mathbf{A}) \Big]$$

$$\mathbf{K}_i \;=\; +\,m_i \Big[-(\vec{a}_i - \vec{A}\,) + 2\,\vec{\omega} \times (\vec{v}_i - \vec{V}\,) - \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] + \vec{\alpha} \times (\vec{r}_i - \vec{R}) \Big]$$

where:

$$\mathbf{E} \;=\; -\,\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} \;=\; \nabla \times \mathbf{A}$$

$$\phi \;=\; -\,{}^{1\!}/_{2}\,[\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,]^2 \;+\; {}^{1\!}/_{2}\,(\vec{v}_i - \vec{V}\,)^2$$

$$\mathbf{A} \;=\; -\,[\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,] + (\vec{v}_i - \vec{V}\,)$$

$$\frac{\partial \mathbf{A}}{\partial t} \;=\; -\,\vec{\alpha} \times (\vec{r}_i - \vec{R}) + (\vec{a}_i - \vec{A}\,)$$

$$\nabla\phi \;=\; \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{R})\,]$$

$$\nabla \times \mathbf{A} \;=\; -\,2\,\vec{\omega}$$

The net kinetic force $\mathbf{K}_i$ acting on a particle i of mass m_i can also be obtained starting from the following kinetic energy:

$$K_i \;=\; -\,m_i\,[\,\phi - (\vec{v}_i - \vec{V}\,) \cdot \mathbf{A}\,]$$

$$K_i \;=\; {}^{1\!}/_{2}\,m_i\,\big[\,(\vec{v}_i - \vec{V}\,) - \vec{\omega} \times (\vec{r}_i - \vec{R})\,\big]^2$$

$$K_i \;=\; {}^{1\!}/_{2}\,m_i\,\big[\,\mathbf{v}_i\,\big]^2$$

Since the kinetic energy K_i must be positive, then applying the following Euler-Lagrange equation, we obtain:

$$\mathbf{K}_i \;=\; -\,\frac{d}{dt}\left[\frac{\partial\,{}^{1\!}/_{2}\,m_i\,\big[\,\mathbf{v}_i\,\big]^2}{\partial\,\mathbf{v}_i}\right] + \frac{\partial\,{}^{1\!}/_{2}\,m_i\,\big[\,\mathbf{v}_i\,\big]^2}{\partial\,\mathbf{r}_i} \;=\; -\,m_i\,\mathbf{a}_i$$

where $\mathbf{r}_i$, $\mathbf{v}_i$ and $\mathbf{a}_i$ are the invariant position, the invariant velocity and the invariant acceleration of particle i (see Annex II)

Appendix B

Fields and Potentials II

The net kinetic force $\mathbf{K}_i$ acting on a particle i of mass m_i (relative to a reference frame S fixed to a particle s ($\vec{r}_s = \vec{v}_s = \vec{a}_s = 0$) of mass m_s, with invariant velocity $\mathbf{v}_s$ and invariant acceleration $\mathbf{a}_s$) can also be expressed as follows:

$$\mathbf{K}_i \;=\; + \, m_i \Big[\, \mathbf{E} + \vec{v}_i \times \mathbf{B} \, \Big]$$

$$\mathbf{K}_i \;=\; + \, m_i \Big[- \nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + \vec{v}_i \times (\nabla \times \mathbf{A}) \Big]$$

$$\mathbf{K}_i \;=\; + \, m_i \Big[- (\vec{a}_i + \mathbf{a}_s) + 2 \, \vec{\omega} \times \vec{v}_i - \vec{\omega} \times (\vec{\omega} \times \vec{r}_i \,) + \vec{\alpha} \times \vec{r}_i \; \Big]$$

where:

$$\mathbf{E} \;=\; - \, \nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} \;=\; \nabla \times \mathbf{A}$$

$$\phi \;=\; - \, {}^1\!/_2 \, (\vec{\omega} \times \vec{r}_i \,)^2 \; + \; {}^1\!/_2 \, (\vec{v}_i + \mathbf{v}_s)^2$$

$$\mathbf{A} \;=\; - \, (\vec{\omega} \times \vec{r}_i \,) + (\vec{v}_i + \mathbf{v}_s)$$

$$\frac{\partial \mathbf{A}}{\partial t} \;=\; - \, \vec{\alpha} \times \vec{r}_i \; + (\vec{a}_i + \mathbf{a}_s)$$

$$\nabla\phi \;=\; \vec{\omega} \times (\vec{\omega} \times \vec{r}_i \,)$$

$$\nabla \times \mathbf{A} \;=\; - \, 2 \, \vec{\omega}$$

The net kinetic force $\mathbf{K}_i$ acting on a particle i of mass m_i can also be obtained starting from the following kinetic energy:

$$K_i \;=\; - \, m_i \, [\, \phi - (\vec{v}_i + \mathbf{v}_s) \cdot \mathbf{A} \,]$$

$$K_i \;=\; {}^1\!/_2 \, m_i \, \big[\, (\vec{v}_i + \mathbf{v}_s) - (\vec{\omega} \times \vec{r}_i \,) \, \big]^2$$

$$K_i \;=\; {}^1\!/_2 \, m_i \, \big[\, \mathbf{v}_i \, \big]^2$$

Since the kinetic energy K_i must be positive, then applying the following Euler-Lagrange equation, we obtain:

$$\mathbf{K}_i \;=\; - \, \frac{d}{dt} \left[\frac{\partial \, {}^1\!/_2 \, m_i \, \big[\, \mathbf{v}_i \, \big]^2}{\partial \, \mathbf{v}_i} \right] + \frac{\partial \, {}^1\!/_2 \, m_i \, \big[\, \mathbf{v}_i \, \big]^2}{\partial \, \mathbf{r}_i} \;=\; - \, m_i \, \mathbf{a}_i$$

where $\mathbf{r}_i$, $\mathbf{v}_i$ and $\mathbf{a}_i$ are the invariant position, the invariant velocity and the invariant acceleration of particle i (see Annex II)

Appendix C

Fields and Potentials III

The kinetic force $\mathbf{K}_{ij}$ exerted on a particle i of mass m_i by another particle j of mass m_j can also be expressed as follows:

$$\mathbf{K}_{ij} \;=\; +\, m_i\, m_j\, M^{-1} \Big[\, \mathbf{E} + (\vec{v}_i - \vec{v}_j) \times \mathbf{B} \,\Big]$$

$$\mathbf{K}_{ij} \;=\; +\, m_i\, m_j\, M^{-1} \Big[-\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + (\vec{v}_i - \vec{v}_j) \times (\nabla \times \mathbf{A}) \Big]$$

$$\mathbf{K}_{ij} \;=\; +\, m_i\, m_j\, M^{-1} \Big[-(\vec{a}_i - \vec{a}_j) + 2\,\vec{\omega} \times (\vec{v}_i - \vec{v}_j) - \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{r}_j)\,] + \vec{\alpha} \times (\vec{r}_i - \vec{r}_j) \Big]$$

where:

$$\mathbf{E} \;=\; -\,\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

$$\mathbf{B} \;=\; \nabla \times \mathbf{A}$$

$$\phi \;=\; -\,{}^1\!/_2\,[\,\vec{\omega} \times (\vec{r}_i - \vec{r}_j)\,]^2 \;+\; {}^1\!/_2\,(\vec{v}_i - \vec{v}_j)^2$$

$$\mathbf{A} \;=\; -\,[\,\vec{\omega} \times (\vec{r}_i - \vec{r}_j)\,] \;+\; (\vec{v}_i - \vec{v}_j)$$

$$\frac{\partial \mathbf{A}}{\partial t} \;=\; -\,\vec{\alpha} \times (\vec{r}_i - \vec{r}_j) + (\vec{a}_i - \vec{a}_j)$$

$$\nabla\phi \;=\; \vec{\omega} \times [\,\vec{\omega} \times (\vec{r}_i - \vec{r}_j)\,]$$

$$\nabla \times \mathbf{A} \;=\; -\,2\,\vec{\omega}$$

The kinetic force $\mathbf{K}_{ij}$ exerted on a particle i of mass m_i by another particle j of mass m_j can also be obtained starting from the following kinetic energy:

$$K_{ij} \;=\; -\, m_i\, m_j\, M^{-1} \big[\, \phi - (\vec{v}_i - \vec{v}_j) \cdot \mathbf{A} \,\big]$$

$$K_{ij} \;=\; {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, (\vec{v}_i - \vec{v}_j) - \vec{\omega} \times (\vec{r}_i - \vec{r}_j) \,\big]^2$$

$$K_{ij} \;=\; {}^1\!/_2\, m_i\, m_j\, M^{-1} \big[\, \mathbf{v}_i - \mathbf{v}_j \,\big]^2$$

Since the kinetic energy K_{ij} must be positive, then applying the following Euler-Lagrange equation, we obtain:

$$\mathbf{K}_{ij} \;=\; -\,\frac{d}{dt} \left[\frac{\partial\, {}^1\!/_2\, \frac{m_i\, m_j}{M} \big[\, \mathbf{v}_i - \mathbf{v}_j \,\big]^2}{\partial\, [\, \mathbf{v}_i - \mathbf{v}_j \,]} \right] + \frac{\partial\, {}^1\!/_2\, \frac{m_i\, m_j}{M} \big[\, \mathbf{v}_i - \mathbf{v}_j \,\big]^2}{\partial\, [\, \mathbf{r}_i - \mathbf{r}_j \,]} \;=\; -\,\frac{m_i\, m_j}{M} \big[\, \mathbf{a}_i - \mathbf{a}_j \,\big]$$

where $\mathbf{r}_i, \mathbf{v}_i, \mathbf{a}_i, \mathbf{r}_j, \mathbf{v}_j$ and $\mathbf{a}_j$ are the invariant positions, the invariant velocities and the invariant accelerations of particles i and j (see Annex II)