Some Results on the Greatest Common Divisor of Two Integers

$(bx + cy, b) = (cy, b)$ for all integers x and y.
Proof.
Since $(bx+cy, b) \mid (bx+cy)$ and $(bx+cy, b) \mid bx$, $(bx+cy, b) \mid ((bx+cy)-bx)$. Thus, $(bx+cy, b) \mid cy$. Moreover, $(bx + cy, b) \mid b$. Thus,

$$(bx + cy, b) \mid (cy, b). \tag{1}$$

Since $(cy, b) \mid bx$ and $(cy, b) \mid cy$, $(cy, b) \mid (bx + cy)$. Also, $(cy, b) \mid b$. Therefore,

$$(cy, b) \mid (bx + cy, b) \tag{2}$$

So

$$(bx + cy, b) = (cy, b) \tag{3}$$

by (1) and (2).

It can be shown by the similar method as above that

$$(bx + cy, c) = (bx, c). \tag{4}$$

If $(b, c) = 1$ then $(a, bc) = (a, b)(a, c)$.
Proof.
There exist integers m_0 and n_0 such that $(a, b) = m_0a + n_0b$. Similarly, $(a, c) = ma + nc$ for some integers m and n. Since $(a, bc) \mid a$ and $a \mid ((m_0a)(ma) + (m_0a)(nc) + (n_0b)(ma))$,

$$(a, bc) \mid ((m_0a)(ma) + (m_0a)(nc) + (n_0b)(ma)). \tag{5}$$

Moreover, $(a, bc) \mid bc$ and $bc \mid (n_0b)(nc)$. So

$$(a, bc) \mid (n_0b)(nc). \tag{6}$$

Hence $(a, bc) \mid ((m_0a)(ma) + (m_0a)(nc) + (n_0b)(ma) + (n_0b)(nc))$ by (5) and (6). In other words,

$$(a, bc) \mid (a, b)(a, c). \tag{7}$$

Since $(b, c) = 1$, $m'b+n'c = 1$ for some integers m' and n'. Since $(a, b) \mid b$ and $(a, c) \mid a$, $(a, b)(a, c) \mid ba$. It follows that

$$(a, b)(a, c) \mid m'ba. \tag{8}$$

Also $(a, b) \mid a$ and $(a, c) \mid c$. Thus $(a, b)(a, c) \mid ca$. So

$$(a, b)(a, c) \mid n'ca. \tag{9}$$

From (8) and (9), $(a, b)(a, c) \mid (m'ba + n'ca)$. In other words,

$$(a, b)(a, c) \mid a. \tag{10}$$

Since $(a, b) \mid b$ and $(a, c) \mid c$,

$$(a, b)(a, c) \mid bc. \tag{11}$$

From (10) and (11),

$$(a,b)(a,c) \mid (a,bc). \tag{12}$$

So

$$(a,bc) = (a,b)(a,c). \tag{13}$$

by (7) and (12).

If $(b,c) = 1$ then $(cy,b)(bx,c) = (b,y)(c,x)$ for all integers x and y.
Proof.
Since $(b,c) = 1$, $mb + nc = 1$ for some integers m and n. Since $(cy,b) \mid mb$ and $(cy,b) \mid cy$, $(cy,b) \mid (mby + ncy)$. So $(cy,b) \mid y$. Since $(cy,b) \mid b$ and $(cy,b) \mid y$,

$$(cy,b) | (b,y). \tag{14}$$

Since $(bx,c) \mid bx$ and $(bx,c) \mid cx$, $(bx,c) \mid (mbx + ncx)$. So $(bx,c) \mid x$. Since $(bx,c) \mid c$ and $(bx,c) \mid x$,

$$(bx,c) \mid (c,x). \tag{15}$$

Hence

$$(cy,b)(bx,c) \mid (b,y)(c,x) \tag{16}$$

by (14) and (15).

Since $(b,y) \mid cy$ and $(b,y) \mid b$,

$$(b,y) \mid (cy,b). \tag{17}$$

Since $(c,x) \mid bx$ and $(c,x) \mid c$,

$$(c,x) \mid (bx,c). \tag{18}$$

Hence

$$(b,y)(c,x) \mid (cy,b)(bx,c) \tag{19}$$

by (17) and (18).
Therefore

$$(cy,b)(bx,c) = (b,y)(c,x) \tag{20}$$

by (16) and (19).

If $(b,c) = 1$ then $(bx + cy, bc) = (b,y)(c,x)$ for all integers x and y.
Proof.

$$\begin{aligned} (bx+cy,bc) &= (bx+cy,b)(bx+cy,c) && \text{by (13)} \\ &= (cy,b)(bx,c) && \text{by (3) and (4)} \\ &= (b,y)(c,x) && \text{by (20)} \end{aligned}$$