Number of Pythagorean Triples and Expansion of Euclid's formula

Aryan Phadke

Abstract

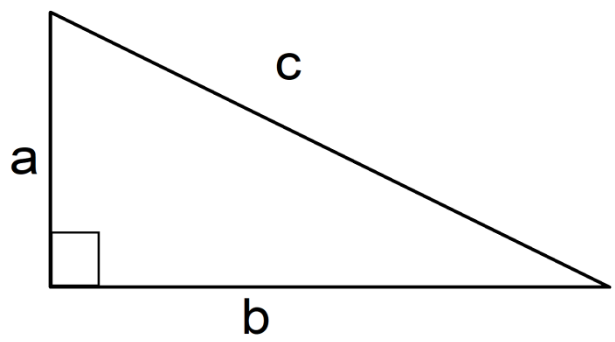
Set of Pythagorean triple consists of three values such that they comprise the three sides of a right angled triangle. Euclid gave a formula to find Pythagorean Triples for any given number. Motive of this paper is to find number of possible Pythagorean Triples for a given number. I have been able to provide a different proof for Euclid's formula, as well as find the number of triples for any given number. Euclid's formula is altered a little and is expanded with a variable 'x'. When 'x' follows the conditions mentioned the result is always a Pythagorean Triple.

Introduction

- Introduction to Pythagorean Triples

Pythagorean Triple is a set of three integers such that they comprise the three sides of a right angled triangle.

Let {a, b, c} be a Pythagorean triple where[$a \leq b < c$]. Then



So, by the Pythagorean Theorem, we conclude that $[a^2 + b^2 = c^2]$

Eg. If it is given that $[a = 5]$ then there are infinite number of combinations for the values of 'b' and 'c' that follow the Pythagorean Theorem.

If then it is added that $[b = 12]$ then the number of combinations is reduced to a single combination.

$$\left[c = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13\right]$$

The set $\{a, b, c\} = \{5, 12, 13\}$ will be a Pythagorean triple.

- Euclid's Formula to find Triples
  - For any two arbitrary pair of integers *m* and *n,* with $m > n > 0$. Integers, $(a = m^2 - n^2), (b = 2mn), (c = m^2 + n^2)$ form a Pythagorean triple.

- Aim and Introduction of the Expansion
  - Euclid's formula is based on any integers (m,n). In this paper, we will see an altered Euclid's formula, assuming that any one value of the triple is given. This way, we can find Pythagorean Triple for a specific number, rather than integers (m,n).
  - The Altered formula provides not only one but multiple Pythagorean triples where the given number is the Hypotenuse or a side other than the hypotenuse.
  - Proof of the expansion and alteration lies within the series of squares of integers. The logic that difference between squares of two consecutive integers is an odd integer is fundamental.

Result

Formula for a side and Hypotenuse

If the value of a side other than the hypotenuse is given, such as [a], then

$\left[b = \frac{a^2 - x^2}{2x}\right]$ and $\left[c = \frac{a^2 + x^2}{2x}\right]$

Where 'x' has to follow three conditions

1) $[x \leq a]$
2) $if\ [a \in odd\ integer]\ then\ [x\ \in odd\ integer]$
   $if\ [a \in even\ integer]\ then\ [x \in even\ integer]$.
3) $[x = {a^2}/{n}]$

   $where\ (x, n, a \in odd\ integers)\ or\ (x, n, a \in even\ integers)$

Eg. Let $[a = 15]$

1) $[x \leq a]$. So, $[x = \{1,2,3,4,5, \ldots \ldots \ldots \ldots \ldots \ldots \ldots 15\}]$
2) $x\ \in odd\ integer$. So, $[x = \{1, 3, 5, 7, 9, 11, 13, 15\}]$.
3) $\left[x = {a^2}/{n}\right]$. So, $[x = 1,3,5,9,15]$

- If we consider $(x = 3)$ then
  $\left[b = \frac{15^2-3^2}{2\times3} = \frac{225-9}{6} = \frac{216}{6} = 36\right]$
  $\left[c = \frac{15^2+3^2}{2\times3} = \frac{225+9}{6} = \frac{234}{6} = 39\right]$
  So, the integer Pythagorean Triple when $(a = 15)\ and (x = 3)$ is
  $\{a, b, c\} = \{15, 36, 39\}$

- ## Formula for sides other than Hypotenuse

  If the value of the hypotenuse is given i.e. [c] then
  $\left[a = \sqrt{2cx - x^2}\right]$ and $[b = c - x]$

  Where 'x' has to follow three conditions

  1) $\left[x < \frac{C}{\sqrt{2}(\sqrt{2}+1)}\right]$
  2) $[x\ \in Integers]$
  3) $\left[x = \frac{2C}{(n^2+1)}\right]$ where $[n \in natural]$.

- E.g. Let $[c = 75]$

1) $\left[x < \frac{C}{\sqrt{2}(\sqrt{2}+1)}\right]$ So, $[x < 21.96699141]$
2) $[x \in Integers]$ So, $[x = \{1, 2, 3, 4, 5, \ldots \ldots \ldots \ldots \ldots \ldots, 21\}]$

3) $\left[x = \frac{2C}{(n^2+1)} = \frac{150}{(n^2+1)}\right]$ So, $[x = \{3, 15\}]$

- If we consider $(x = 3)$ then
  $\left[a = \sqrt{2 \times 75 \times 3 - 3^2} = \sqrt{441} = 21\right]$
  $[b = (75 - 3) = 72]$
  So, the integer Pythagorean triple when $(c = 75)$ and $(x = 3)$ is
  $\{a, b, c\} = \{21, 72, 75\}$

Discussion

Proof

- ### Fundamentals for the proof

1. Series of squares of natural numbers.
   $1, 4, 9, 16, 25, 36, 49, 64, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots, n^2$
2. Difference between two consecutive squares is an odd number.
   Eg. $[13^2 - 12^2 = 169 - 144 = 25]$
3. Difference between two squares of integers with a difference of 'x' is the sum of 'x' number of consecutive odd numbers.
   Eg. $[13^2 - 10^2 = 169 - 100 = 69]$
   $[(13^2 - 12^2) + (12^2 - 11^2) + (11^2 - 10^2)]$
   $[(169 - 144) + (144 - 121) + (121 - 100)]$
   $[25 + 23 + 21]$
   Since we know that difference between consecutive squares is an odd number, then equation above will be a sum of 'x' consecutive odd numbers.

- ### Formula to find the adjacent side and the hypotenuse

'a' is given, and difference between 'c' and 'b' is equal to 'x'.
So, $(b = c - x)$
According to Pythagorean Theorem,
$c^2 - b^2 = a^2$
$= c^2 - (c - x)^2 = c^2 - (c^2 - 2cx + x^2)$

$= 2cx - x^2 = a^2$
$= 2cx = a^2 + x^2$
$\therefore c = \dfrac{a^2 + x^2}{2x}$
$b = c - x = \dfrac{a^2 + x^2}{2x} - x$
$\therefore b = \dfrac{a^2 + x^2 - 2x^2}{2x} = \dfrac{a^2 - x^2}{2x}$
$\left[b = \frac{a^2-x^2}{2x}\right]$ and $\left[c = \frac{a^2+x^2}{2x}\right]$

- Formula to find the adjacent sides of an hypotenuse

  'c' is given, and difference between 'c' and 'b' is equal to 'x'.
  So, $(b = c - x)$
  We use the formula from above.
  $c = \dfrac{a^2 + x^2}{2x}$
  $2cx = a^2 + x^2$
  $a^2 = 2cx - x^2$
  $\therefore a = \sqrt{2cx - x^2}$
  $\left[a = \sqrt{2cx - x^2}\right]$ and $[b = c - x]$

Formula Analysis

- Valid values of 'x' for $\{a, b, c\} \in integers$, when 'a' is given.
  - Rule 1

    $b = \frac{a^2-x^2}{2x}$, since 'b' is a positive integer,
    $a^2 \geq x^2$
    $\therefore Rule\ 1$
    $[x \leq a]$

  - Rule 2

    $a^2 = c^2 - (c - x)^2$
    So, using the third fundamental, we can conclude that
    $a^2 = sum\ of\ 'x' consecutive\ odd\ integers.$

If $a \in odd\ integers$, then $a^2 \in odd\ integers$.
So, $sum\ of\ 'x'consecutive\ odd\ integers\ \in odd\ integers$.
Since we know that sum of odd number of odd numbers is an odd number. Eg. (3+5+7+9+11=35)
$\therefore x \in odd\ integers$.
Similarly, if $a \in even\ integers$, then $x \in even\ integers$.
$\therefore Rule\ 2$.
$[(if\ a \in odd\ integers, then\ x \in odd\ integers)]$.
$[(if\ a \in even\ integers, then\ x \in even\ integers)]$.

- Rule 3

Since $b \in integers$,

$$\frac{a^2 - x^2}{2x} \in integers,$$

$$\frac{a^2}{2x} - \frac{x}{2} \in integers,$$

As we know that, both $a, x\ \in odd\ or\ even\ integers$.
When divided by 2, both 'x' and 'a' result in either multiple of '1' or a multiple of '0.5'.

$$\therefore \frac{a^2}{x} - x\ \in integers.$$

$$\therefore \frac{a^2}{x} \in integers$$

$x = \frac{a^2}{n}$, where $n \in integers$.
Let, $(a \in even\ integers)$, then $(a^2 \in even\ integers)$.
By Rule 2, $(x \in even\ integers)$.

$$a^2 = x \times n$$

$$b = \frac{a^2 - x^2}{2x} = \frac{xn - x^2}{2x} = \frac{n - x}{2} = \frac{n}{2} - \frac{x}{2}$$

Since, $(x \in even)$ then for $b \in integers$
$(n \in even\ integers)$.
$\therefore Rule\ 3 =$

$$\left[x = {a^2}/{n}\right]$$

$(x, n, a \in even\ integers)$ or $(x, n, a \in odd\ integers)$.

- Valid values of ‘x’ for $\{a, b, c\} \in integers$, when ‘c’ is given.
  - Rule 1

    Since, $(a < b)$,

    $\sqrt{2cx - x^2} < (c - x)$

    Squaring both sides, we get

    $2cx - x^2 < (c^2 - 2cx + x^2)$

    $\therefore c^2 - 4cx + 2x^2 > 0$

    By Solving the Quadratic equation, we get

    $$c > \frac{4x \pm \sqrt{16x^2 - 8x^2}}{2} = 2x \pm x\sqrt{2}$$

    $(c > 2x + x\sqrt{2})$ or $(c > 2x - x\sqrt{2})$

    $\left(c > x\sqrt{2}(\sqrt{2} + 1)\right)$ or $\left(c > x\sqrt{2}(\sqrt{2} - 1)\right)$

    Since the latter is lesser than former,

    $c > x\sqrt{2}(\sqrt{2} + 1)$

    $\therefore Rule\ 1$

    $$\left[x < \frac{c}{\sqrt{2}(\sqrt{2} + 1)}\right]$$

  - Rule 2

    $(b \in integers)$.

    $\therefore (c - x) \in integers$

    $As, (c \in integers)$

    $\therefore Rule\ 2$

    $[x \in integers]$.

  - Rule 3

    When the value of ‘c’ is given, interchanging the values of ‘a’ and ‘b’ would result only in one combination of a Pythagorean Triple. So, when ‘a’ < ‘b’ would yield the same result as when ‘b’ < ‘a’.

    Assuming ‘a’ < ‘b’

    $$a < \frac{a^2 - x^2}{2x}$$

    $= 2ax < a^2 - x^2$

    $= x^2 + 2ax - a^2 < 0$

By solving the Quadratic equation, we get

$$x < \frac{-2a \pm \sqrt{4a^2 + 4a^2}}{2} = -a \pm a\sqrt{2}$$

Since $x > 0$

$x < a(\sqrt{2} - 1)$

So we can conclude that, $\left(x < \frac{a}{2}\right)$.

As we know that, a^2 *is divisble by* x

And $\left(x < {}^{a}/_{2}\right)$.

$'a'$ *will also be divisble by* $'x'$.

$\therefore$ $[a = x \times n]$ where $n \in$ *integers*.

$xn = \sqrt{2cx - x^2}$

Squaring both sides, we get

$x^2 \times n^2 = 2cx - x^2$

$= x \times n^2 = 2c - x$

$2c = xn^2 + x$

$2c = x(n^2 + 1)$

$\therefore$ *Rule* 3

$$\left[x = \frac{2c}{(n^2 + 1)}\right].$$

- Number of Pythagorean Triples

  No. of integer Pythagorean Triples = valid values of $'x'$.

Eg. Let the given number be 75.

For ‘75’ to be a side other than the hypotenuse.

I. $[x \leq a]$

   $\therefore [x \leq 75]$

II. $[(if\ a \in odd\ integers, then\ x \in odd\ integers)]$.

   $[(if\ a \in even\ integers, then\ x \in even\ integers)]$.

   $\therefore [x \in odd] \therefore [x = \{1,3,5,7,9,11, \dots \dots \dots \dots \dots, 75\}]$

III. $\left[x = {}^{a^2}/_{n}\right]$

   $(x, n, a \in even\ integers)$ or $(x, n, a \in odd\ integers)$.

$\therefore \left[x = {}^{5625}/_{n}\right] \therefore [x = \{1,3,5,9,15,25,45,75\}]$

$(No. of\ Integer\ Pythagorean\ Triples = 8)$

For '75' to be the hypotenuse.

I. $\left[x < \frac{c}{\sqrt{2}(\sqrt{2}+1)}\right]$

$\therefore \left[x < \frac{75}{\sqrt{2}(\sqrt{2}+1)}\right] \therefore [x < 21.96699141]$

II. $[x \in integers]$.

$\therefore [x = \{1,2,3,4,5,6,7, \dots \dots \dots \dots \dots \dots .., 21\}$

III. $\left[x = \frac{2c}{(n^2+1)}\right]$

$\therefore \left[x = {}^{150}/_{(n^2+1)}\right] \therefore [x = \{3,15\}]$

$(No. of\ Integer\ Pythagorean\ Triples = 2)$

$[Total\ no. of\ Integer\ Pythagorean\ Triples\ consisting\ '75' = 10]$

Formula verification

- Formula when side other than the hypotenuse is given.

Let $(a = 20)$

Then,

1. $[x \leq a]$

   $\therefore [x \leq 20]$
2. $[(if\ a \in odd\ integers, then\ x \in odd\ integers)]$.

   $[(if\ a \in even\ integers, then\ x \in even\ integers)]$.

   $\therefore [x \in even\ integers] \therefore [x = \{2,4,6,8,10, \dots \dots \dots \dots, 20\}]$
3. $\left[x = {}^{a^2}/_{n}\right]$

   $(x, n, a \in even\ integers)$ or $(x, n, a \in odd\ integers)$.

   $\left[x = {}^{400}/_{n}\right] \therefore [x = \{2,4,8,10,20\}]$

➢ $(x = 2)$ and $(a = 20)$ then,

$\left(b = \frac{20^2-2^2}{2\times2} = \frac{396}{4} = 99\right)$ and $\left(c = \frac{20^2+2^2}{2\times2} = \frac{404}{4} = 101\right)$

- $(x = 4)$ and $(a = 20)$ then,

  $\left(b = \frac{20^2-4^2}{2\times4} = \frac{384}{8} = 48\right)$ and $\left(c = \frac{20^2+4^2}{2\times4} = \frac{416}{8} = 52\right)$
- $(x = 8)$ and $(a = 20)$ then,

  $\left(b = \frac{20^2-8^2}{2\times8} = \frac{336}{16} = 21\right)$ and $\left(c = \frac{20^2+8^2}{2\times8} = \frac{463}{16} = 29\right)$
- $(x = 10)$ and $(a = 20)$ then,

  $\left(b = \frac{20^2-10^2}{2\times10} = \frac{300}{20} = 15\right)$ and $\left(c = \frac{20^2+10^2}{2\times10} = \frac{500}{20} = 25\right)$
- $(x = 20)$ and $(a = 20)$ then,

  $\left(b = \frac{20^2-20^2}{2\times20} = \frac{0}{40} = 0\right)$ and $\left(c = \frac{20^2+20^2}{2\times20} = \frac{800}{40} = 20\right)$

- Formula when the hypotenuse is given.

Let $(c = 100)$

Then,

1. $\left[x < \frac{c}{\sqrt{2}(\sqrt{2}+1)}\right]$

   $\therefore \left[x < \frac{100}{\sqrt{2}(\sqrt{2}+1)}\right] \therefore [x < 29.28932188]$
2. $[x \in integers]$.

   $\therefore [x = \{1,2,3,4,5,6,7, \ldots \ldots \ldots \ldots \ldots \ldots \ldots.,29\}]$
3. $\left[x = \frac{2c}{(n^2+1)}\right]$

   $\therefore \left[x = {}^{200}/_{(n^2+1)}\right] \therefore [x = \{4,20\}]$

- $(x = 4)$ and $(c = 100)$

  $\left(a = \sqrt{2 \times 100 \times 4 - 4^2} = \sqrt{784} = 28\right)$ and $(b = 96)$
- $(x = 20)$ and $(c = 100)$

  $\left(a = \sqrt{2 \times 100 \times 20 - 20^2} = \sqrt{3600} = 60\right)$ and $(b = 80)$

- Formula for finding the number of Pythagorean Triples

Let the given number = m = 50

- If 'm' is a side other than the hypotenuse

  Then,

1. $[x \leq m]$
   $\therefore [x \leq 50]$
2. $[(if\ m \in odd\ integers, then\ x \in odd\ integers)]$.
   $[(if\ m \in even\ integers, then\ x \in even\ integers)]$.
   $\therefore [x \in even\ integers] \therefore [x = \{2,4,6,8,10, \dots \dots \dots \dots ,50\}]$
3. $\left[x = {m^2}/_{n}\right]$
   $(x, n, a \in even\ integers)$ or $(x, n, a \in odd\ integers)$.
   $\left[x = {2500}/_{n}\right] \therefore [x = \{2,10,50\}]$
   $(No.of\ Pythagorean\ Triples = 3)$

- If 'm' is the hypotenuse
  Then,

1. $\left[x < \frac{m}{\sqrt{2}(\sqrt{2}+1)}\right]$
   $\therefore \left[x < \frac{50}{\sqrt{2}(\sqrt{2}+1)}\right] \therefore [x < 14.64466094]$
2. $[x \in integers]$.
   $\therefore [x = \{1,2,3,4,5,6,7, \dots \dots \dots \dots \dots \dots \dots . ,14\}$
3. $\left[x = \frac{2m}{(n^2+1)}\right]$
   $\therefore \left[x = {100}/_{(n^2+1)}\right] \therefore [x = \{2,10\}]$
   $(No.of\ Pythagorean\ Triples = 2)$

- $(Total\ no.of\ Pythagorean\ Triples = 3 + 2 = 5)$

Conclusion

- The Formula devised in the paper is a different interpretation of Euclid's formula for finding Pythagorean Triples. Euclid's formula is based on two integer (m,n) and its implementation on any side of the triangle. While this formula is based on the assumption that one side is given and focuses on finding multiple combinations fitting the other two sides.

- Formula devised in the paper presents multiple Pythagorean Triples while the original formula resulted into only one. Unlike other formulas of its kind, this formula also provides us with the number of possible combinations of Pythagorean Triples when any one side is given.
- Basic step in using Euclid's formula is interpreting the given number as $(m \times n)$ where, *m* and *n* are integers. This will not always be the case.
  Eg. $(a = 15)$. If we assume $(a = mn)$
  Then $(a = 1 \times 15), (a = 3 \times 5)$.
  So according to Euclid's formula
  Number of Pythagorean triples = 4.
  But by the formula devised in this paper,
  Number of Pythagorean triples = 6.

Future Research and Potential

- With this expansion of Euclid's formula, we now can find every Pythagorean Triple present. So, the first Diophantine equation can be solved
- By using the same logic from this paper's proof, we can implement them to other Diophantine equations and try to solve those equations.
- Similar execution can be used to find the formula to find Pythagorean Quadruples.
- This formula can also be used as a tool for other results. The problems related to finding triples or number of triples can easily be tackled.

References

1) *Heath, Thomas (1956), The Thirteen Books of Euclid's Elements Vol. 1 (Books I and II) (2nd ed.), Dover Publications, ISBN 978-0-486-60088-8*
2) *Alperin, Roger C. (2005), "The modular tree of Pythagoras" (PDF), American Mathematical Monthly, **112** (9): 807–816, CiteSeerX 10.1.1.112.3085, doi:10.2307/30037602, JSTOR 30037602, MR 2179860*
3) *Berggren, B. (1934), "Pytagoreiska trianglar", Tidskrift för Elementär Matematik, Fysik och Kemi (in Swedish), **17**: 129–139*
4) *Barning, F.J.M. (1963), "Over pythagorese en bijna-pythagorese driehoeken en een generatieproces met behulp van unimodulaire matrices" (PDF), Math. Centrum Amsterdam Afd. Zuivere Wisk. (in Dutch), ZW-011: 37*