Pi in Terms of Phi

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Abstract. We give two formulas for Pi in terms of Phi.

1. Introduction

1.1. The number ϕ (Phi) is defined by

$$\phi = \frac{1+\sqrt{5}}{2} \tag{1}$$

Some relations

$$\phi^2 = \phi + 1 \tag{2}$$

$$\phi = \sqrt{1+\sqrt{1+\sqrt{1+\ldots}}} \tag{3}$$

1.2. The number Pi is defined by

$$\pi = 4\int_0^1 \frac{1}{1+x^2}\,dx = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 3.141592\ldots \tag{4}$$

2. Pi in terms of Phi

$$\pi = 2\tan^{-1}\left(\frac{1}{2\phi}\right) + 4\sum_{n=0}^{\infty} \frac{\phi^{-2n-1}}{2n+1} \sum_{k=0}^{n} \binom{2n+1}{2k+1} (-1)^k \phi^{-(n-k)} \tag{5}$$

$$\pi = 2\tan^{-1}\left(\frac{1}{2\sqrt{\phi}}\right) + \frac{4}{\sqrt{\phi}}\sum_{n=0}^{\infty} \frac{(-1)^n \phi^{-2n-1}}{2n+1} \sum_{k=0}^{n} \binom{2n+1}{2k} (-1)^k \phi^{-(n-k)} \tag{6}$$

The formulas (5),(6) can be expressed as

$$\pi = 2\tan^{-1}\left(\frac{1}{2\phi}\right) + 4\sum_{n=0}^{\infty} \frac{\phi^{-2n-1}}{2n+1} \operatorname{Im}\left(\left(\frac{1}{\sqrt{\phi}} + i\right)^{2n+1}\right) \tag{7}$$

$$\pi = 2\tan^{-1}\left(\frac{1}{2\sqrt{\phi}}\right) + 4\sum_{n=0}^{\infty} \frac{(-1)^n \phi^{-2n-1}}{2n+1} \operatorname{Re}\left(\left(\frac{1}{\sqrt{\phi}} + i\right)^{2n+1}\right) \tag{8}$$

where $i = \sqrt{-1}$, and $\operatorname{Re}(z), \operatorname{Im}(z)$ are the real and imaginary part of z .

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