Refutation of a 'concrete' Rauszer Boolean algebra generated by a preorder

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Abstract: From the 11 equations tested, we refute 13 artifacts:

1. a condition for "an existential quantifier ∃ … on a Boolean algebra";
2. "a quantifier ∃ as closure operator on B, for which every open element is closed";
3. the interior operator on abstract topological Boolean algebra;
4. the kernel of a homomorphism from a Heyting algebra into another as a filter;
5. deductive systems and filters as equivalent;
6. the atomic definition of p ≤ ∃p in Halmos algebra;
7. a 'concrete' Rauszer Boolean algebra;
8. two conditions for the definition of a filter (and Heyting algebra using the filter);
9. a De Morgan algebra as a Kleene algebra;
10. equivalences of symmetrical Heyting algebras;
11. equivalences in Heyting algebras;
12. intuitionistic implication of intuitionistic logic; and
13. a theorem and a proposition of Nelson algebras.

As a result, the following seven areas are *non* tautologous fragments of the universal logic VŁ4:

1. Topological Boolean algebra;
2. Heyting algebra;
3. Intuitionistic logic;
4. Halmos algebra;
5. Rauszer algebra;
6. Kleene algebra; and
7. Nelson algebra.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·; \ Not And;
> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥, zero;
(%z>#z) N as non-contingency, Δ, ordinal 1; (%z<#z) C as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: Iturrioz, L. (2019). About a 'concrete' Rauszer Boolean algebra generated by a preorder. arxiv.org/pdf/1905.09928.pdf luisa.iturrioz@math.univ-lyon1.fr

(page 2) Recall that […] an existential quantifier ∃ ... on a Boolean algebra (B,∧,∨,−,0,1) is a mapping ∃∶B→B satisfying the following conditions:

(∃0) ∃0 = 0 (0.1.1)

LET p, q: a, b; (p=p) 1 or T; (p@p) 0 or **F**.

%(p@p)=(p@p) ; NNNN NNNN NNNN NNNN (0.1.2)

Remark 0.1.2: Eq. 0.1.2 is not tautologous, thereby refuting a condition for "an existential quantifier ∃ … on a Boolean algebra".

(∃1) a ∧ ∃a = a (1.1.1)

Remark 1.1.1: Eq. 1.1.1 as a tautology seems to be another misstatement in the literature of the Halmos algebra. For example the equivalent as (Q_2):

From: Halmos, P.R. (1954). Algebraic logic, I. Monadic boolean algebras. *Compositio Mathematica*, tome 2 (1954-1956). 217-249. numdam.org/article/CM_1954-1956__12__217_0.pdf [image]

(Q_2) p ≤ ∃p, page 4 (Q2.1)

~(%p<p) = (p=p) ; NTNT NTNT NTNT NTNT (Q2.2)

Remark Q2.2: Eq. Q2.2 as rendered refutes the Halmos algebra at its most atomic level.

[T]he image ∃(B) (i.e. the range of the quantifier ∃), is a monadic Boolean subalgebra of B. In addition, x∈∃(B) if and only if ∃x=x, if and only if ∀x=x. (2.1)

LET p, q: a, B

((#p=p)>(%p=p))>(p<%q) ; CNC**F** CNC**F** CNC**F** CNC**F** (2.2)

An element x such that ∃x = x (resp. ∀x = x) is called closed (resp. open), constant or a fixpoint, and the set of closed elements is the same as the set of open elements [...]. In other words, a quantifier ∃ is a closure operator on B, for which every open element is closed.

Remark 2.2: Eq. 2.2 is not tautologous, refuting "a quantifier ∃ is a closure operator on B, for which every open element is closed".

2. A 'concrete' Rauszer Boolean algebra

[B]ased on semisimplicity motivations, A. Monteiro [...], has studied properties of several binary operations in abstract topological Boolean algebras (A, I), where A is a Boolean algebra and I is an interior operator on A. In particular, he dealt with an implication ⇒ [...], [where ⊃ is the classical implication x ⊃ y = −x ∪ y] defined by:

a ⇒ b = I(Ia ⊃ Ib) (2.3.1)

(p>q)=(r&((r&p)>(r&q))) ; **F**T**FF** TTTT **F**T**FF** TTTT (2.3.2)

Remark 2.3.2: Eq. 2.3.2 is *not* tautologous, and so refutes the interior operator on abstract topological Boolean algebra.

3. Representation theorems in an unified form

For the sake of clarity we recall that a subset F of a lattice $(A, \wedge, \vee, 0, 1)$ is said to be a filter if the following conditions are satisfied:

(f1) $1 \in F$; (3.1.1.1)

LET p, q, r, s, t: P, Q, a, b, A, F; (p=p) 1 or `T`.

`(p=p)<t ;` `TTTT TTTT TTTT TTTT,`
`FFFF FFFF FFFF FFFF` (3.1.1.2)

(f2) if $a, b \in F$, then $a \wedge b \in F$; (3.1.2.1)

Remark 3.2.1: Eq. 3.2.1 is a trivial tautology.

(f3) if $a \in F$ and $a \leq b$, then $b \in F$; (3.1.3.1)

`((r<t)&~(s<r))>(s<t) ;` `TTTT` **`FFFF`** `TTTT TTTT,`
`TTTT TTTT TTTT TTTT` (3.1.3.2)

We note, incidentally, that for Heyting algebras, the kernel of a homomorphism from a Heyting algebra into another, is a filter. Also, the notions of deductive systems and filters are equivalent (A. Monteiro, 1959).

Remark 3: Two conditions for a filter are not tautologous, thereby refuting the definition of a filter and Heyting algebra which uses the filter.

B) Representation of symmetrical Heyting algebras

A De Morgan algebra A [is a Kleene algebra for]

($K_{a,b}$) $a \wedge \sim a \leq b \vee \sim b$, for any $a,b \in A$ holds (3.2.1)

`(#(r&s)<t)>~((s+~s)<(r&~r)) ;`
`TTTT TTTT TTTT CCCC,`
`TTTT TTTT TTTT TTTT` (3.2.2)

Remark 3.2.2: Eq. 3.2.2 is not tautologous, thereby refuting a De Morgan algebra as a Kleene algebra.

Definition 3.4

[T]he following equivalences on account of the intuitionistic equality $x \wedge (x \Rightarrow y) = x \wedge y$ [...]:
(3.4.1.1)

LET p, q: x, y

(p&(p>q))=(p+q) ;   T**FF**T T**FF**T T**FF**T T**FF**T   (3.4.1.2)

Remark 3.4.1.2: Eq. 3.4.1.2 is *not* tautologous, thereby refuting the intuitionistic equality as claimed.

Theorem 3.5 Proof: equivalences

P∈h(~x)⇔~x∈P⇔x∈~P⇔x∉−(~P)=ϕ(P)⇔ϕ(P)∉h(x)⇔P∉ϕ(h(x))⇔P∈−ϕ(h(x))=~h(x).
(3.5.1)

LET p, q, r, s: P, h, x, ϕ

(((p<(q&~r))=((~r<p)=(r<~p)))=(~(r<~(~p=(s&p)))=~((s&p)<(q&r))))=((~(p<(s&(q&r)))=(p <(~s&(q&r))))=(~q&r)) ;   TTT**F** T**FFF** T**F**TT T**FF**T   (3.5.2)

Remark 3.5.2: Eq. 3.5.2 is not tautologous, thereby refuting equivalences of symmetrical Heyting algebras.

C) Representation of Nelson algebras
Theorem 3.8 Proof:

[(a∧b)⇒(~a∨~b∨c)]≤a⇒[~a∨(b⇒(~b∨c))]   (3.8.1.1)

~(p<((p&q)>(~p+(~q+r))))>(~p+(q>(~q+r))) ;   (3.8.1.2)

By the definition of the intuitionistic implication this is equivalent to

a∧[(a∧b)⇒(~a∨~b∨c)]≤~a∨[b⇒(~b∨c)]   (3.8.2.1)

~((~p+(q>(~q+r)))<(p&((p&q)>(~p+(~q+r))))) = (p=p)

Remark 3.8: Eqs 3.8.1.1 = 3.8.2.1   (3.8.3.1)

(~(p<((p&q)>(~p+(~q+r))))>(~p+(q>(~q+r))))=
~((~p+(q>(~q+r)))<(p&((p&q)>(~p+(~q+r))))) ;
FT**F**T **F**T**F**T **F**T**F**T **F**T**F**T   (3.8.3.2)

Eq. 3.8.3.2 is *not* tautologous as claimed, hence refuting Theorem 3.8 as a representation of Nelson algebras.

Proposition 3.9

LET   p, q, r: G, H, K

(G∩H)⇒(~G∪~H∪K)⊆G⇒(~G∪(H⇒(~H∪K)))   (3.9.1.1)

(p&q)>(~(p<(~p+(~q+r)))>(~p+(q>(~q+r)))) ;   (3.9.1.2)

By the definition of the intuitionistic implication this is equivalent to

G∩[(G∩H)⇒(~G∪~H∪K)]⊆~G∪(H⇒(~H∪K))   (3.9.2.1)

~((~p+(q>(~q+r)))<(p&((p&q)>(~p+(~q+r))))) = (p=p) ; (3.9.2.2)

Remark 3.9: Eqs 3.8.1.1 = 3.8.2.1 (3.9.3.1)

((p&q)>(~(p<(~p+(~q+r)))>(~p+(q>(~q+r)))))=
~((~p+(q>(~q+r)))<(p&((p&q)>(~p+(~q+r))))) ;
FT**F**T **F**T**F**T **F**T**F**T **F**T**F**T (3.9.3.2)

Eq. 3.9.3.2 is *not* tautologous as claimed, hence refuting a proposition of Nelson algebras.

From the 11 equations tested, we refute 13 artifacts:

1. a condition for "an existential quantifier ∃ … on a Boolean algebra";
2. "a quantifier ∃ as closure operator on B, for which every open element is closed";
3. the interior operator on abstract topological Boolean algebra;
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As a result, the following are seven fields are *non* tautologous fragments of universal logic VŁ4: topological Boolean algebra; Heyting algebra; intuisionistic logic; Halmos algebra; Rauszer algebra; Kleene algebra; and Nelson algebra.