Refutation of Presburger arithmetic via Axiom 2

Abstract: In Presburger arithmetic, Axiom 2 as *x*+1 = *y*+1 → *x*=*y* is *not* tautologous. Therefore Presburger arithmetic is a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓ ,·; \ Not And;
\> Imply, greater than, →, ⇒ , ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇔, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) N as non-contingency, Δ, ordinal 1; (%z<#z) C as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y), (x ⊑ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: en.wikipedia.org/wiki/Presburger_arithmetic

The language of Presburger arithmetic contains constants 0 and 1 and a binary function +, interpreted as addition. In this language, the axioms of Presburger arithmetic are the universal closures of the following:

[Axiom] 2. *x*+1 = *y*+1 → *x*=*y* (2.1)

LET p, q: x, y; (%r>#r) 1; (r=r) T

((p+(%r>#r))=(q+(%r>#r)))>(p=q) ;
TCCT TCCT TCCT TCCT (2.2)

Remark 2.2: If Eq. 2.1 takes ordinal constant 1 as T, then:

((p+(r=r))=(q+(r=r)))>(p=q) ;
T**FF**T T**FF**T T**FF**T T**FF**T (2.3)

Remark 2.1: We attempt to resuscitate Eq. 2.1 by removing 1 from the antecedent:

[(*x*+1 = *y*+1) -1] → *x*=*y* (3.1)

(((p+(%r>#r))=(q+(%r>#r)))-(%r>#r))>(p=q) ;
TNNT TNNT TNNT TNNT (3.2)

Eqs. 2.2, 2.3, and 3.2 are not tautologous, thereby refuting Presburger arithmetic by its own Axiom 2.