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Abstract

We proof Goldbach's Conjecture. We use results obtained by Srinivāsa A. Rāmānujan (specifically in his paper A Proof of Bertrand's Postulate). A generalization of the conjecture is also proven for every natural not coprime with a natural $m > 1$ and greater or equal than $2m$.

Notation and observations

- Natural numbers

  $\mathbb{N}$ denotes the set of natural numbers, and all of them are here greater than zero (positive integers).

- Primorial:

  It is denoted by $p#$ the primorial of the prime number $p$, meaning the product of all primes less or equal than $p$.

- Omission of subjects:

  To speed up the writing, we will omit sometimes the word number following others as prime, composite, even, odd, natural, coprime, since they are obvious here.

- In what follows, $p_i$ denotes the $i$-th prime.
Preliminary

The Bertrand’s postulate also known as Bertrand-Chebyshev theorem, was proposed by Joseph Bertrand in 1845 [1] and proven by Pafnuti L. Chebyshëv in 1850 [2], [5], Srinivāsa A. Rāmānujan in 1919 [3] and Paul Erdős in 1932 [4].

Here we will use the following result, obtained by Ramanujan’s in [3], after his proof of Bertrand’s postulate:

**Theorem 1. (Ramanujan)**

Let \( \pi(x) \) denote the number of primes not exceeding \( x \). Then \( \pi(x) - \pi(x/2) \geq 1, 2, 3, 4, 5, \ldots \), if \( x \geq 2, 11, 17, 29, 41, \ldots \) respectively.

**PROOF.** See [3].

The Bertrand’s postulate follows from the first case: if \( x \geq 2 \), then \( \pi(x) - \pi(x/2) \geq 1 \). Assigning to \( x \) a natural number \( n > 1 \), we have that \( \pi(n) - \pi(n/2) \geq 1 \) and we can write:

For every natural \( n > 1 \) there exists at least one prime number \( p \) such that \( n/2 < p \leq n \).

We will use here the less restrictive statement:

\[
\text{For every } n > 1 \text{ there exists a prime number } p \text{ such that } n/2 \leq p \leq n \tag{01}
\]

**Results**

**Lemma.**

Let’s be \( i \geq 1 \) and \( \ell \geq 2 \), natural numbers.

If \( \ell \) is coprime with \( p_i\# \) and \( p_{i+1} \leq \ell < (p_{i+1})^2 \) then \( \ell \) is prime.

**PROOF.**

We prove that the smallest composite coprime with \( p_i\# \) greater than \( p_{i+1} \) is \( (p_{i+1})^2 \).

Since \( p_{i+1} \) is the prime immediately after \( p_i \), \( p_{i+1} \) cannot be a divisor of \( p_i\# \). Likewise, none of the the powers of \( p_{i+1} \) share any divisors with \( p_i\# \). The smallest of this powers greater than \( p_{i+1} \) is \( (p_{i+1})^2 \), consequently \( (p_{i+1})^2 \) is a composite number actually being coprime with \( p_i\# \). Note that \( (p_{i+1})^2 \) is coprime with \( p_i\# \) because \( (p_{i+1})^2 = (p_{i+1}) (p_{i+1}) \) and \( p_i\# = p_1 p_2 \ldots p_i \); the only prime divisor of \( (p_{i+1})^2 \) is \( p_{i+1} \), and as \( p_{i+1} > p_j \) for all \( j = 1, 2, \ldots, i \), the prime \( p_{i+1} \) cannot divide \( p_i\# \).

Any other coprime with \( p_i\# \) smaller than \( (p_{i+1})^2 \) and greater or equal than \( p_{i+1} \) will necessarily be a prime, because if it were a composite, it would be divisible by one of the \( p_1, p_2, \ldots, p_i \), or by \( p_{i+1} \); in the first case, it would contradict that it is coprime with \( p_i\# \) and in the second case there is only the possibility of being \( p_{i+1} \), which is a prime and therefore contradicts the hypothesis of being composite.

QED
Theorem 2.

Given a natural \( n > 1 \), there exists two primes \( p, q \), such that \( 2n = p + q \)

PROOF.

We will prove that for any natural \( n > 1 \), there exist one prime \( p \) such that \( 2n - p \) is also prime. Writing then \( q = 2n - p \), the statement of the theorem is obtained.

- The theorem is verified for every natural \( n \leq 6 \), since it has been shown to be true for all numbers less than \( 4 \times 10^{18} \) (Oliveira e Silva, 2013), see [6].

- For any natural \( n > 6 \), by application of (01), we have that there exists at least one prime \( p \) that verifies \( n/2 \leq p \leq n \).

Since \(-1 + (8n + 5)^{1/2} < n \) for all \( n > 6 \), we can say that for every natural \( n > 6 \), there exists at least one prime, be \( p_{i+1} \) for some natural \( i > 1 \), which verifies the inequalities:

\[
\frac{-1 + (8n + 5)^{1/2}}{2} \leq p_{i+1} \leq n, \quad (02)
\]

from where it follows, operating:

From the first inequality, \((8n + 5)^{1/2} \leq 2p_{i+1} + 1\); squaring, \(8n + 5 \leq 4(p_{i+1})^2 + 4p_{i+1} + 1\); rearranging, \(8n - 4p_{i+1} \leq 4(p_{i+1})^2 - 4\), and simplifying, \(2n - p_{i+1} \leq (p_{i+1})^2 - 1\).

From the second inequality, \(p_{i+1} \leq n\), then \(2p_{i+1} \leq 2n\) and \(p_{i+1} \leq 2n - p_{i+1}\), so we have:

\[
p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1 \quad (03)
\]

Note that since \(p_{i+1} \leq n\), the maximum value for the index \( i \) must be \( \pi(n) - 1 \). Then we can write:

\[
\text{For every natural } n > 6, \text{ there exists a natural } i \text{ with } 1 < i < \pi(n), \text{ such that } p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1, \quad (04)
\]

if \(2n - p_{i+1}\) is coprime with \(p_i\), by application of the previous lemma it will be necessary prime.

The proof concludes by demonstrating that:

\[
\text{For every natural } n > 6, \text{ there exists a natural } i \text{ with } 1 < i < \pi(n), \text{ such that } (2n - p_{i+1} \text{ is coprime with } p_i) \text{ and } (p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1) \quad (05)
\]

Note that for all natural \( n > 6 \), always exists at least one prime \( p_{i+1} \), for some \( i > 1 \), verifying (03), as we showed before.

We proceed by reduction to absurdity. The reduction hypothesis consists in supposing that:

\[
\text{There exists a natural } n > 6, \text{ such that for all natural } i \text{ with } 1 < i < \pi(n), \text{ it's not true that: } (2n - p_{i+1} \text{ is coprime with } p_i) \text{ and } (p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1) \quad (06)
\]
The statement \((2n - p_{i+1} \text{ is coprime with } p_i \#)\) in (06) implies \((p_{i+1} \leq 2n - p_{i+1})\), because if \(2n - p_{i+1} < p_{i+1}\), then the number \(2n - p_{i+1}\) is necessarily not coprime with \(p_i \#\), since in such a case, it must be a prime less or equal to \(p_i\) or to a number divisible by at least some of the primes \(p_j\) with \(1 < j \leq i\). Applying this to (06), we can write:

There exists a natural \(n > 6\), such that for all natural \(i\) with \(1 < i < \pi(n)\), it’s not true that:

\[
(2n - p_{i+1} \text{ is coprime with } p_i \#) \text{ and } (2n - p_{i+1} \leq (p_{i+1})^2 - 1 )
\]  

(07)

As by (04), for all natural \(n > 6\) there exists a natural \(i\) with \(1 < i < \pi(n)\), such that 

\(2n - p_{i+1} \leq (p_{i+1})^2 - 1\); then, also applying the logic rule De Morgan’s law of negation of conjunction, the statement (07) can be written as:

There exists a natural \(n > 6\), such that for all natural \(i\) with \(1 < i < \pi(n)\),

\(2n - p_{i+1} \) is not coprime with \(p_i \#\)

(08)

If \(2n - p_{i+1} \) is not coprime with \(p_i \#\) for some natural \(i\) with \(1 < i < \pi(n)\), there exists a natural \(j\), with \(1 \leq j \leq i\), such that \(p_j \mid p_i \#\) and \(p_j \mid (2n - p_{i+1})\).

For \(i = 2\), be \(j\) a natural with \(1 \leq j \leq 2\). If \(p_j \mid 5\# = 30\). Then, \(j = 1\) or \(j = 2\), i.e., \(p_j = 2\) or \(p_j = 3\) and \(p_j \mid 30\) in both cases. We have that \(2 \mid (2n - 5)\) since \((2n - p_{i+1})\) is odd, thus \(3 \mid (2n - 5)\) and there exists a natural number \(k\) such that \(2n - 5 = 3k\), or, rearranging,

\[
2n = 5 + 3k
\]  

(09)

For \(i = 3\), be \(j\) a natural with \(1 \leq j \leq 3\). Now \(j\) can take the values 1, 2 or 3. As we saw before, 2 cannot divide the odd number \((2n - 7)\), then \(j = 2\) or \(j = 3\). If \(j = 2\), reasoning in the same way as in obtaining (09), we have that there exists a natural \(\ell\) such that:

\[
2n = 7 + 3\ell
\]  

(10)

Equating the expressions for \(2n\) (09) and (10), we have:

\[
3k - 3\ell = 2
\]  

(11)

According to Bézout’s identity, the Diophantine equation (11) has an integer solution if and only if \(\gcd(3, -3)\) is a divisor of 2. As this does not happen, since \(\gcd(3, -3) = 3 \nmid 2\), therefore equation (11) has no natural solution.

If \(j = 3\), using the same process, we have that there exists a natural \(m\) such that:

\[
2n = 7 + 5m
\]  

(12)

Equating the expressions for \(2n\) (09) and (12), we have:

\[
3k - 5m = 2
\]  

(13)

According to Bézout’s identity, we also conclude that equation (13) has no solution.

A contradiction is reached, so that the reduction hypothesis leads to an absurdity and the starting proposition (05) is true.

QED
**Theorem 3.** Generalization of the Goldbach’s Theorem.

For \( m > 1 \), every not coprime with \( m \) greater or equal than \( 2m \) can be written as the sum of \( m \) primes.

**PROOF.**

Let \( n \) be a natural not coprime with \( m \), with \( n \geq 2m \), for some \( m > 1 \).

For \( m = 2 \), the proposition is the Goldbach’s Conjecture, previously proven.

For \( m > 2 \):

- If \( n/m = p \in \mathbb{P} \), then \( n = mp \) and \( n \) is expressed as the sum of \( m \) primes \( p \).

- If \( n/m \not\in \mathbb{P} \), then we can write \( n = 2m + r \), for any natural \( r \).

  - If \( r \) is even, then \( r + 4 \) is also even and we can write: \( n = 2(m - 2) + (r + 4) \). Applying Theorem 2 to the even number \( r + 4 \), there exist two primes \( p, q \), such that \( r + 4 = p + q \). Therefore we can write \( n = 2(m - 2) + p + q \), which is the sum of \( m \) primes, since \( 2(m - 2) \) is equal to \( (m - 2) \) times the sum of the prime number 2.

  - If \( r \) is odd, then we write: \( n = 2(m - 3) + r + 6 = 2(m - 3) + 3 + (r + 3) \). Thus, being odd \( r \), the number \( (r + 3) \) is even and applying Theorem 2 can be written as the sum of two primes, \( r + 3 = p + q \). Thus, \( n = 2(m - 3) + 3 + p + q \), which is, as above, the sum of \( m \) primes.

**QED**

**Corollary 1.**

Let \( m, n > 1 \) two natural numbers. Then \( m \times n \) can be written as the sum of \( m \) primes.

**PROOF.**

The natural number \( n \times m \) is greater than 1, is not coprime with \( m \) and is greater or equal than \( 2m \). We apply the Theorem 3 to the numbers \( n \times m \) and \( m \).

**QED**

**Corollary 2.**

Let \( m, n > 1 \) two natural numbers. Then \( m^n \) can be written as the sum of \( m \) primes.

**PROOF.**

By applying the Theorem 3 to the numbers \( m^n \) and \( m \).

**QED**
References


