About Goldbach's Conjecture

Jesús Esteve
Jorge Enrique Martínez
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Authors

Jesús Esteve Pérez. (Spain).

Taller de Acción Audiovisual, S.L.

editorial.eb@gmail.com

with the collaboration of Jorge Enrique Martínez. (Argentina).

Abstract

We proof Goldbach's Conjecture. We use results obtained by *Srinivāsa A. Rāmānujan* (specifically in his paper *A Proof of Bertrand's Postulate*). A generalization of the conjeture is also proven for every natural not coprime with a natural $m > 1$ and greater or equal than $2m$.

Notation and observations

- Natural numbers

  N denotes the set of natural numbers, and all of them are here greater than zero (positive integers).

- Primorial:

  It is denoted by $p\#$ the primorial of the prime number p, meaning the product of all primes less or equal than p.

- Omission of subjects:

  To speed up the writing, we will omit sometimes the word *number* following others as *prime, composite, even, odd, natural, coprime,* since they are obvious here.

- In what follows, p_i denotes the i-th prime.

Preliminary

The Bertrand's postulate also known as Bertrand-Chebyshev theorem, was proposed by Joseph Bertrand in 1845 [1] and proven by Pafnuti L. Chebyshëv in 1850 [2], [5], Srinivāsa A. Rāmānujan in 1919 [3] and Paul Erdős in 1932 [4].

Here we will use the following result, obtained by Ramanujan's in [3], after his proof of Bertrand's postulate:

Theorem 1. (Ramanujan)

Let $\pi(x)$ denote the number of primes not exceeding x. Then $\pi(x) - \pi(x/2) \geq 1, 2, 3, 4, 5, \ldots$, if $x \geq 2, 11, 17, 29, 41, \ldots$ respectively.

PROOF. See [3].

The Bertrand's postulate follows from the first case: if $x \geq 2$, then $\pi(x) - \pi(x/2) \geq 1$. Assigning to x a natural number $n > 1$, we have that $\pi(n) - \pi(n/2) \geq 1$ and we can write:

For every natural $n > 1$, there exists at least one prime number p such that $n/2 < p \leq n$.

We will use here the less restrictive statement:

$$\text{For every } n > 1 \text{ there exists a prime number } p \text{ such that } n/2 \leq p \leq n \qquad (01)$$

Results

Lemma.

Let's be $i \geq 1$ and $\ell \geq 2$, natural numbers.

$$\text{If } \ell \text{ is coprime with } p_i\# \text{ and } p_{i+1} \leq \ell < (p_{i+1})^2 \text{ then } \ell \text{ is prime.}$$

PROOF.

We prove that the smallest composite coprime with $p_i\#$ greater than p_{i+1} is $(p_{i+1})^2$.

Since p_{i+1} is the prime inmediately after p_i, p_{i+1} cannot be a divisor of $p_i\#$. Likewise, none of the the powers of p_{i+1} share any divisors with $p_i\#$. The smallest of this powers greater than p_{i+1} is $(p_{i+1})^2$, consecuently $(p_{i+1})^2$ is a composite number actually being coprime with $p_i\#$. Note that $(p_{i+1})^2$ is coprime with $p_i\#$ because $(p_{i+1})^2 = (p_{i+1})\,(p_{i+1})$ and $p_i\# = p_1 p_2 \ldots p_i$; the only prime divisor of $(p_{i+1})^2$ is p_{i+1}, and as $p_{i+1} > p_j$ for all $j = 1, 2, \ldots, i$, the prime p_{i+1} cannot divide $p_i\#$.

Any other coprime with $p_i\#$ smaller than $(p_{i+1})^2$ and greater or equal than p_{i+1} will necessarily be a prime, because if it were a composite, it would be divisible by one of the $p_1, p_2, \ldots, p_i$ or by p_{i+1}; in the first case, it would contradict that it is coprime with $p_i\#$ and in the second case there is only the possibility of being p_{i+1}, wich is a prime and therefore contradicts the hypothesis of being composite.

QED

Theorem 2.

Given a natural $n > 1$, there exists two primes p, q, such that $2n = p + q$

PROOF.

We will prove that for any natural $n > 1$, there exist one prime p such that $2n - p$ is also prime. Writing then $q \equiv 2n - p$, the statement of the theorem is obtained.

• The theorem is verified for every natural $n \leq 7$, since it has been shown to be true for all number less than 4×10^{18} (*Oliveira e Silva, 2013*), see [6].

• For any natural $n > 7$, by application of (01), we have that there exists at least one prime p that verifies $n/2 \leq p \leq n$.

Since $-1 + (8n + 5)^{1/2} < n$ for all $n > 7$, we can say that for every natural $n > 7$, there exists at least one prime, be p_{i+1} for some natural $i > 1$, wich verifies the inequalities:

$$\frac{-1 + (8n + 5)^{1/2}}{2} \leq p_{i+1} \leq n, \qquad (02)$$

from where it follows, operating:

From the first inequality, $(8n + 5)^{1/2} \leq 2p_{i+1} + 1$; squaring, $8n + 5 \leq 4(p_{i+1})^2 + 4p_{i+1} + 1$; rearranging, $8n - 4p_{i+1} \leq 4(p_{i+1})^2 - 4$, and simplifying, $2n - p_{i+1} \leq (p_{i+1})^2 - 1$.

From the second inequality, $p_{i+1} \leq n$, then $2p_{i+1} \leq 2n$, and $p_{i+1} \leq 2n - p_{i+1}$, so we have:

$$p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1 \qquad (03)$$

Then we can write:

For every natural $n > 7$, there exists a natural $i > 1$ such that $p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1$,	(04)

if $2n - p_{i+1}$ is coprime with $p_i\#$, by application of the previous lemma it will be necessary prime.

The proof concludes by demonstrating that:

For every natural $n > 7$, there exists a natural $i > 1$ such that $2n - p_{i+1}$ is coprime with $p_i\#$ and $p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1$.	(05)

Note that for all natural $n > 7$, allways exists at least one prime p_{i+1}, for some $i > 1$, verifying (03), as we showed before.

We proceed by reduction to absurdity. The reduction hypothesis consists in supposing that:

There exists a natural $n > 7$, such that for all natural $i > 1$, it's not true that: ($2n - p_{i+1}$ is coprime with $p_i\#$) and ($p_{i+1} \leq 2n - p_{i+1}$) and ($2n - p_{i+1} \leq (p_{i+1})^2 - 1$)	(06)

The statement ($2n - p_{i+1}$ is coprime with $p_i\#$) implies ($p_{i+1} \leq 2n - p_{i+1}$), since if $2n - p_{i+1} < p_{i+1}$, then the number $2n - p_{i+1}$ is necessarily not coprime with $p_i\#$, since in such a case, it must be a prime less or equal to p_i or a number divisible by at least any of the primes p_j, with $1 < j \leq i$. Applying this to (06), we can write:

There exists a natural $n > 7$, such that for all natural $i > 1$, it's not true that: $(p_{i+1} \leq 2n - p_{i+1})$ and $(2n - p_{i+1} \leq (p_{i+1})^2 - 1)$	(07)

Unifying the two inequalities in only one, we obtain (03):

There exists a natural $n > 7$, such that for all natural $i > 1$, it's not true that: $p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1$	(08)

but (08) contradicts (04) (not only (05)). As we saw before, in the deduction of theese inequalities, for any $n > 7$ and some $i > 1$, $p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1$ if Ramanujan's theorem (01) is true.

A contradiction is reached, so that the reduction hypothesis leads to an absurdity and the starting proposition is true.

QED

Theorem 3. *Generalization of the Goldbach's Theorem.*

For $m > 1$, every not coprime with m greater or equal than $2m$ can be written as the sum of m primes.

PROOF.

Let n be a natural not coprime with m, with $n \geq 2m$, for some $m > 1$.

For $m = 2$, the proposition is the *Goldbach's Conjecture*, previously proven.

For $m > 2$:

- If $n/m = p \in \mathbf{P}$, then $n = mp$ and n is expressed as the sum of m primes p.

- If $n/m \notin \mathbf{P}$, then we can write $n = 2\,m + r$, for any natural r.

  - If r is even, then $r + 4$ is also even and we can write: $n = 2\,(m - 2) + (r + 4)$. Applying Theorem 2 to the even number $r + 4$, there exist two primes p, q, such that $r + 4 = p + q$. Therefore we can write $n = 2\,(m - 2) + p + q$, wich is the sum of m primes, since $2\,(m - 2)$ is equal to $(m - 2)$ times the sum of the prime number 2.

  - If r is odd, then we write: $n = 2\,(m - 3) + r + 6 = 2\,(m - 3) + 3 + (r + 3)$. Thus, being odd r, the number $(r + 3)$ is even and applying Theorem 2 can be written as the sum of two primes, $r + 3 = p + q$. Thus, $n = 2\,(m - 3) + 3 + p + q$, wich is, as above, the sum of m primes.

QED

Corollary.

Let $m, n > 1$ two natural numbers. Then $m \times n$ can be writen as the sum of m primes.

PROOF.

The natural number $n \times m$ is greater than 1, is not coprime with m and is greater or equal than $2m$. We apply the Theorem 3 to the numbers $n \times m$ and m.

QED

Corollary 2.

Let $m, n > 1$ two natural numbers. Then m^n can be writen as the sum of m primes.

PROOF.

By applying the Theorem 3 to the numbers m^n and m.

QED

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