

About Goldbach's Conjecture

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Abstract

We proof Goldbach's Conjecture. We use results obtained by *Srinivāsa A. Rāmānujan* (specifically in his paper *A Proof of Bertrand's Postulate*). A generalization of the conjeture is also proven for every natural not coprime with a natural $m > 1$ and greater or equal than $2m$.

Notation and observations

- Natural numbers

\mathbf{N} denotes the set of natural numbers, and all of them are here greater than zero (positive integers).

- Primorial:

It is denoted by $p\#$ the primorial of the prime number p , meaning the product of all primes less or equal than p .

- Omission of subjects:

To speed up the writing, we will omit sometimes the word *number* following others as *prime*, *composite*, *even*, *odd*, *natural*, *coprime*, since they are obvious here.

- In what follows, p_i denotes the i -th prime.

Preliminary

The Bertrand's postulate also known as Bertrand-Chebyshev theorem, was proposed by Joseph Bertrand in 1845 [1] and proven by Pafnuti L. Chebyshev in 1850 [2], [5], Srinivāsa A. Rāmānujan in 1919 [3] and Paul Erdős in 1932 [4].

Here we will use the following result, obtained by Ramanujan's in [3], after his proof of Bertrand's postulate:

Theorem 1. (Ramanujan)

Let $\pi(x)$ denote the number of primes not exceeding x . Then $\pi(x) - \pi(x/2) \geq 1, 2, 3, 4, 5, \dots$, if $x \geq 2, 11, 17, 29, 41, \dots$ respectively.

PROOF. See [3].

The Bertrand's postulate follows from the first case: if $x \geq 2$, then $\pi(x) - \pi(x/2) \geq 1$. Assigning to x a natural number $n > 1$, we have that $\pi(n) - \pi(n/2) \geq 1$ and we can write:

For every natural $n > 1$, there exists at least one prime number p such that $n/2 < p \leq n$.

We will use here the less restrictive statement:

$$\text{For every } n > 1 \text{ there exists a prime number } p \text{ such that } n/2 \leq p \leq n \quad (01)$$

Results

Lemma.

Let's be $i \geq 1$ and $\ell \geq 2$, natural numbers.

If ℓ is coprime with $p_i\#$ and $p_{i+1} \leq \ell < (p_{i+1})^2$ then ℓ is prime.

PROOF.

We prove that the smallest composite coprime with $p_i\#$ greater than p_{i+1} is $(p_{i+1})^2$.

Since p_{i+1} is the prime immediately after p_i , p_{i+1} cannot be a divisor of $p_i\#$. Likewise, none of the powers of p_{i+1} share any divisors with $p_i\#$. The smallest of this powers greater than p_{i+1} is $(p_{i+1})^2$, consequently $(p_{i+1})^2$ is a composite number actually being coprime with $p_i\#$. Note that $(p_{i+1})^2$ is coprime with $p_i\#$ because $(p_{i+1})^2 = (p_{i+1})(p_{i+1})$ and $p_i\# = p_1 p_2 \dots p_i$; the only prime divisor of $(p_{i+1})^2$ is p_{i+1} , and as $p_{i+1} > p_j$ for all $j = 1, 2, \dots, i$, the prime p_{i+1} cannot divide $p_i\#$.

Any other coprime with $p_i\#$ smaller than $(p_{i+1})^2$ and greater or equal than p_{i+1} will necessarily be a prime, because if it were a composite, it would be divisible by one of the p_1, p_2, \dots, p_i or by p_{i+1} ; in the first case, it would contradict that it is coprime with $p_i\#$ and in the second case there is only the possibility of being p_{i+1} , which is a prime and therefore contradicts the hypothesis of being composite.

QED

Theorem 2.

Given a natural $n > 1$, there exists two primes p, q , such that $2n = p + q$

PROOF.

We will prove that for any natural $n > 1$, there exist one prime p such that $2n - p$ is also prime. Writing then $q \equiv 2n - p$, the statement of the theorem is obtained.

- The theorem is verified for every natural $n \leq 7$, since it has been shown to be true for all number less than 4×10^{18} (Oliveira e Silva, 2013), see [6].
- For any natural $n > 7$, by application of (01), we have that there exists at least one prime p that verifies $n/2 \leq p \leq n$.

Since $-1 + (8n + 5)^{1/2} < n$ for all $n > 7$, we can say that for every natural $n > 7$, there exists at least one prime, be p_{i+1} for some natural $i > 1$, wich verifies the inequalities:

$$\frac{-1 + (8n + 5)^{1/2}}{2} \leq p_{i+1} \leq n, \quad (02)$$

from where it follows, operating:

From the first inequality, $(8n + 5)^{1/2} \leq 2 p_{i+1} + 1$; squaring, $8n + 5 \leq 4 (p_{i+1})^2 + 4 p_{i+1} + 1$; rearranging, $8n - 4 p_{i+1} \leq 4 (p_{i+1})^2 - 4$, and simplifying, $2n - p_{i+1} \leq (p_{i+1})^2 - 1$.

From the second inequality, $p_{i+1} \leq n$, then $2 p_{i+1} \leq 2 n$, and $p_{i+1} \leq 2n - p_{i+1}$, so we have:

$$p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1 \quad (03)$$

and also

$$n \leq \frac{(p_{i+1})^2 + p_{i+1} - 1}{2} \quad (04)$$

Since by (03), $p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1$, if $2n - p_{i+1}$ is coprime with p_{i+1} , by application of the previous lemma it will be necessary prime. The proof concludes by demonstrating that, for every natural $n > 7$, there exists a prime p_{i+1} that verifies (03) and such that $2n - p_{i+1}$ is coprime with p_{i+1} .

We proceed by reduction to absurdity. The reduction hypothesis consists in supposing that:

There exists a natural $n > 7$, such that for all natural $i > 1$,
if $p_{i+1} \leq 2n - p_{i+1} \leq (p_{i+1})^2 - 1$, then $2n - p_{i+1}$ is not coprime with p_{i+1}

(05)

For $i = 2$, $p_i = p_2 = 3$; $p_i^\# = p_2^\# = 6$; $p_{i+1} = p_3 = 5$; applying (04), we have $7 < n \leq (25 + 5 - 1) / 2 = 29 / 2$, therefore $7 < n \leq 14$. There are seven possible values for n : from $n = 8$ to $n = 14$. Then, all possible values for $2n - 5$ are all primes except for $n = 10$ and $n = 13$, where we have $2n - 5 = 15$ and $2n - 5 = 21$, respectively. For $i = 3$, we have, applying (04), that $7 < n \leq 27$; but for $n = 10$, it is $2n - 7 = 13$ and for $n = 13$, it is $2n - 7 = 19$, both primes and therefor not coprimes with $30 = p_3^\#$. The conclusion is that for $i = 2$, there is no possible n to verify the reduction hypothesis. A contradiction is reached, so that the reduction hypothesis leads to an absurdity and the starting proposition is true.

QED

Theorem 3. *Generalization of the Goldbach's Theorem.*

For $m > 1$, every not coprime with m greater or equal than $2m$ can be written as the sum of m primes.

PROOF.

Let n be a natural not coprime with m , with $n \geq 2m$, for some $m > 1$.

For $m = 2$, the proposition is the *Goldbach's Conjecture*, previously proven.

For $m > 2$:

- If $n/m = p \in \mathbf{P}$, then $n = mp$ and n is expressed as the sum of m primes p .

- If $n/m \notin \mathbf{P}$, then we can write $n = 2m + r$, for any natural r .

- If r is even, then $r + 4$ is also even and we can write: $n = 2(m - 2) + (r + 4)$. Applying Theorem 2 to the even number $r + 4$, there exist two primes p, q , such that $r + 4 = p + q$. Therefore we can write $n = 2(m - 2) + p + q$, which is the sum of m primes, since $2(m - 2)$ is equal to $(m - 2)$ times the sum of the prime number 2.

- If r is odd, then we write: $n = 2(m - 3) + r + 6 = 2(m - 3) + 3 + (r + 3)$. Thus, being odd r , the number $(r + 3)$ is even and applying Theorem 2 can be written as the sum of two primes, $r + 3 = p + q$. Thus, $n = 2(m - 3) + 3 + p + q$, which is, as above, the sum of m primes.

QED

Corollary 1.

Let $m, n > 1$ two natural numbers. Then $m \times n$ can be written as the sum of m primes.

PROOF.

The natural number $n \times m$ is greater than 1, is not coprime with m and is greater or equal than $2m$. We apply the Theorem 3 to the numbers $n \times m$ and m .

QED

Corollary 2.

Let $m, n > 1$ two natural numbers. Then m^n can be written as the sum of m primes.

PROOF.

By applying the Theorem 3 to the numbers m^n and m .

QED

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