Approximation of Sum of Harmonic Progression Using Approximation Theory

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Abstract

Background : Sum of Harmonic Progression is an old problem. While a few complex approximations have surfaced, a simple and efficient formula hasn't. Most of the previous formulas use summation as a tool for the approximation. Also, most notable formulas are only applicable to the harmonic progression where the first term and common difference of the corresponding Arithmetic progression is one.

Aim of this paper is to create a formula without using 'summation' as a tool. On the contrary to previous formulas, the resultant formula is applicable to every harmonic progression regardless of the first term and common difference of its corresponding arithmetic progression.

The resultant formula is applicable for numerical values in a harmonic progression. Algebraic terms in a harmonic progression do not value to any numeric terms and for that reason cannot be accounted for the approximation. The resultant formula has a variability in terms of accuracy and complexity to suit different users with different priorities.

The Formula was obtained by using the fundamentals of approximation theory by equating the area enclosed in the graph of harmonic progression and area under the curve of equation (y=1/x). The logic being that the harmonic progression is completely inclusive in the function (y=1/x) is used to formulate the answer.

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1. Introduction

1.1. Sum of Harmonic Progression

Harmonic Progression (HP) is a series of numbers where each term is the reciprocal of the corresponding term of an Arithmetic Progression (AP).

Consider an AP, where

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$$first\,term \;=\; a$$
$$common\,difference \;=\; d$$
$$number\;of\;terms \;=\; n$$
$$last\;term \;=\; L \;=\; a + (n-1)d$$
$$The\;AP \;=\; (a),\;(a+d),\;(a+2d),\;(a+3d),\;.............,\;(L)$$
$$corresponding\;HP \;=\; \tfrac{1}{(a)},\;\tfrac{1}{(a+d)},\;\tfrac{1}{(a+2d)},\;..........\;,\;\tfrac{1}{(L)}$$
$$Sum\;of\;HP \;=\; \tfrac{1}{(a)} + \tfrac{1}{(a+d)} + \tfrac{1}{(a+2d)} + \;..............\; + \tfrac{1}{(L)}$$

1.2. Why Sum of HP is incalculable

It is not possible for a harmonic progression of distinct unit fractions to sum to an integer. The reason being that, necessarily, at least one denominator of the progression will be divisible by a prime number that does not divide any other denominator. [1]

This creates a situation where every different harmonic progression would need a specific formula to be accurate. Thus, we can conclude that no general formula can accurately calculate every type of harmonic progression.

2. Results

Sum of Harmonic Progression = S(L)

$$S(L) \;=\; \tfrac{1}{(a)} + \tfrac{1}{(a+d)} + \tfrac{1}{(a+2d)} + \;..............\; + \tfrac{1}{(L)}$$

Approximation of Sum of HP = f(L)

$$f(L) \;=\; \frac{\ln\left(\dfrac{L}{x}\right)}{d} + \tfrac{1}{2L} + \tfrac{1}{2x} + S(x-d)$$

$$x \;=\; any\;term\;of\;the\;corresponding\;AP$$

$S(x-d) \;=\; \tfrac{1}{(a)} + \tfrac{1}{(a+d)} + \tfrac{1}{(a+2d)} + \;............\; + \tfrac{1}{(x-d)}$ where the term S(x-d) has to be calculated manually.

Formula was obtained by equating the area under the curve of equation (y=1/x) and the area enclosed in the graph of harmonic progression.

Due to the addition of 'x', the accuracy of the formula changes with the change in 'x'. Because S(x-d) has to be calculated manually, the manual work done to calculate the output also increases with the increase in the value of 'x'.

It can also be concluded that

$$\lim_{x\to L} f(L) \;=\; S(L)$$

3. Discussion

3.1. Proof

3.1.1. Graph of Harmonic Progression

3.1.2. Graph of curve of equation (y=1/x)

3.1.3. Area enclosed in Figure 1

The area is equal to sum of areas of all triangles and rectangles.

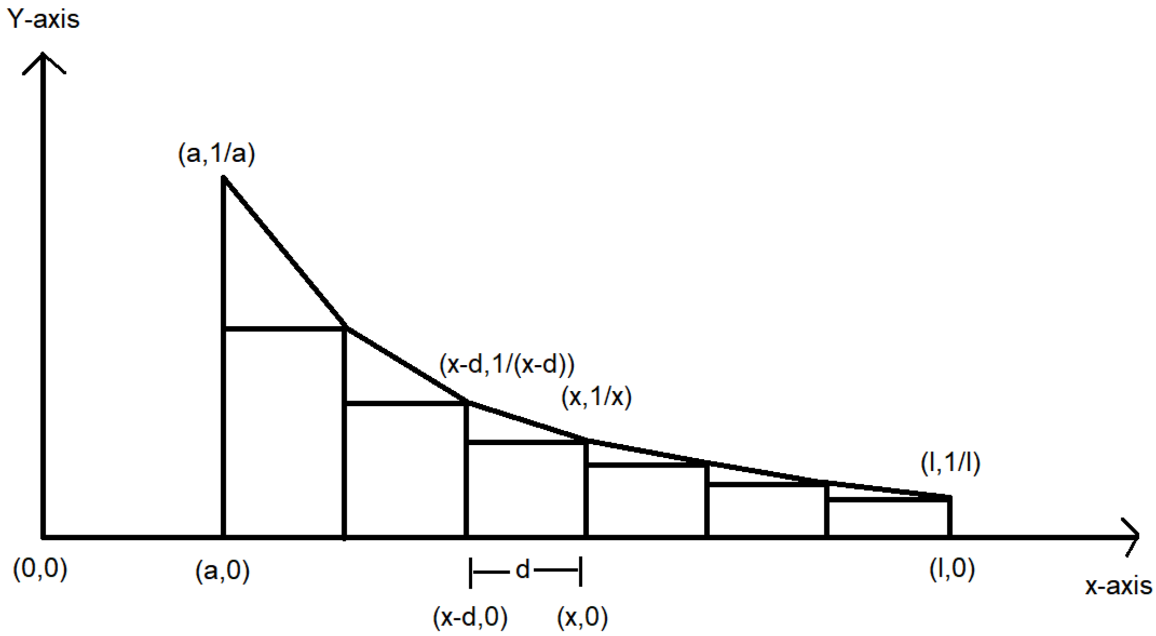

Figure 1: Graph of Harmonic Progression

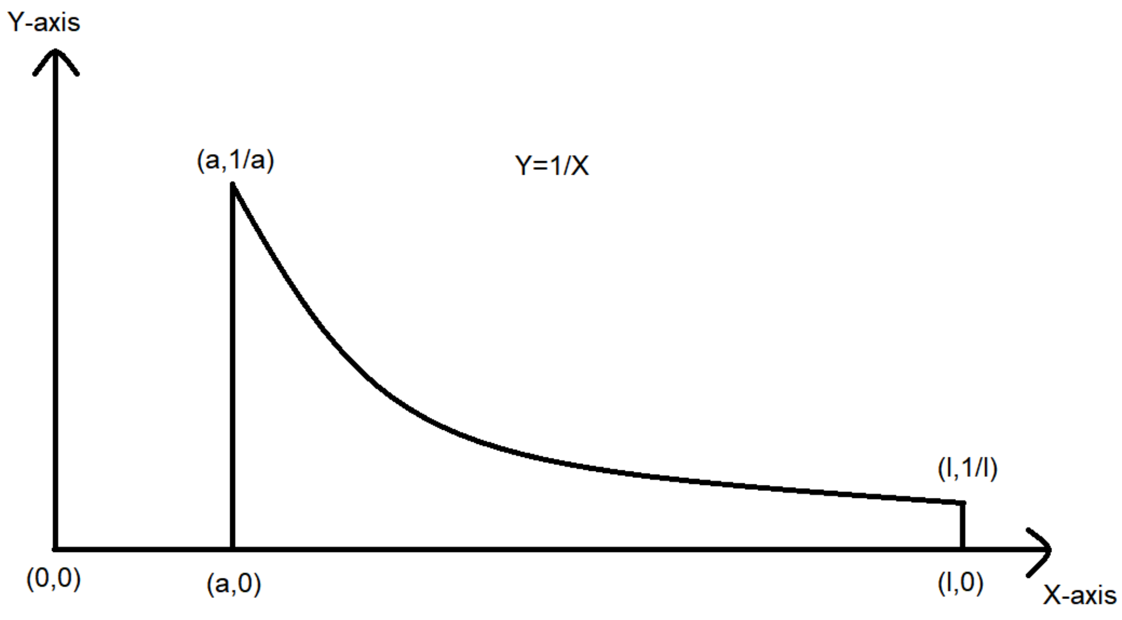

Figure 2: Graph of 'y=1/x'

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Sum of Areas of all Triangles . Area of one triangle = $\frac{1}{2} \times \; base \; \times \; height$
Base for any triangle = d
Height for a triangle = $\left[\frac{1}{(x-d)} - \frac{1}{(x)}\right]$, $[x \in AP]$
Area of one triangle = $\frac{1}{2} \times \; d \; \times \; \left[\frac{1}{(x-d)} - \frac{1}{(x)}\right]$
Sum of Areas of All Triangles = $\sum_{x+(a+d)}^{L} \frac{d}{2} \times \left[\frac{1}{(x-d)} - \frac{1}{(x)}\right]$, $[x \in AP]$

$$= \frac{d}{2} \times \left[\frac{1}{(a)} - \frac{1}{(a+d)} + \frac{1}{(a+d)} - .... - \frac{1}{L}\right]$$

$$= \frac{d}{2} \times \left[\frac{1}{(a)} - \frac{1}{(L)}\right] \tag{1}$$

Sum of Areas of All Rectangles . Area of one rectangle = $base \; \times \; height$
Base for any rectangle = d
Height of a rectangle = $\frac{1}{(x)}$, $[x \in AP]$
Area of one rectangle = $d \times \frac{1}{(x)}$
Sum of Areas of all Rectangles = $\sum_{x=(a+d)}^{L} d \times \frac{1}{(x)}$, $[x \in AP]$

$$= d \times \left[\frac{1}{(a+d)} + \frac{1}{(a+2d)} + \frac{1}{(a+3d)} + ... + \frac{1}{(L)}\right]$$

$$= d \; \times \; \left[S(L) \; - \; \frac{1}{(a)}\right] \tag{2}$$

Area enclosed in Figure 1. Ar(Figure 1) = Equation (1) + Equation (2)
$= \frac{d}{2} \; \times \; \left[\frac{1}{(a)} - \frac{1}{(L)}\right] \; + \; d \; \times \; \left[S(L) - \frac{1}{(a)}\right] = \; d \; \times \; \left[\left(S(L) - \frac{1}{(a)}\right) \; + \; \left(\frac{1}{(2a)} - \frac{1}{(2L)}\right)\right]$

$$= \; d \; \times \; \left[S(L) \; - \; \frac{1}{(2a)} - \; \frac{1}{(2L)}\right] \tag{3}$$

3.1.4. Area enclosed in Figure 2

Area enclosed in Figure 2 is equal to area under the curve of equation (y=1/x)
$= \; \int_a^L y dx \; = \; \int_a^L \frac{1}{(x)} dx$

$$= \; \ln\left(\frac{L}{a}\right) \tag{4}$$

3.1.5. Equating Areas of Figure 1 and Figure 2

We implement the logic that area enclosed in the graph of harmonic progression would be approximately equal to the area under the curve of equation (y=1/x). We can conclude this because of the mutuality between functions of harmonic progression and equation (y=1/x). The only difference between the two functions would be the exclusion of terms between two consecutive terms of the harmonic progression.

Therefore, Equation (3) will approximately be equal to Equation (4)

$d \times \left[S(L) - \frac{1}{(2a)} - \frac{1}{(2L)}\right] \approx \ln\left(\frac{L}{a}\right)$

$S(L) - \frac{1}{(2a)} - \frac{1}{(2L)} \approx \frac{\ln\left(\dfrac{L}{a}\right)}{d}$

$$S(L) \approx \frac{\ln\left(\dfrac{L}{a}\right)}{d} + \frac{1}{(2a)} + \frac{1}{(2L)} \tag{5}$$

3.1.6. Addition of variable 'x'

With the increase in the value of the first term, the difference between slopes of consecutive terms decreases. Thus, we can conclude that when the value of first term increases, the accuracy of the formula will also increase. If 'a' were to be substituted with 'x', for any term of AP then error will decrease.

For a sum of HP $= \frac{1}{(a)} + \frac{1}{(a+d)} + \frac{1}{(a+2d)} + \dots\dots\dots\dots\dots\dots + \frac{1}{(L)}$

We divide this sum into two parts. i.e.

Part 1 $= \frac{1}{(a)} + \frac{1}{(a+d)} + \dots\dots\dots\dots + \frac{1}{(x-d)}$

Part 2 $= \frac{1}{(x)} + \frac{1}{(x+d)} + \frac{1}{(x+2d)} + \dots\dots\dots\dots + \frac{1}{(L)}$

We calculate the value of (Part 2) using the formula in Equation (5)

Part 2 $\approx \frac{\ln\left(\dfrac{L}{x}\right)}{d} + \frac{1}{(2x)} + \frac{1}{(2L)}$

We know that value of (Part 1) is equal to S(x-d)

We add Part 1 to both sides.

$Part\ 1 + Part\ 2 = \frac{\ln\left(\dfrac{L}{x}\right)}{d} + \frac{1}{(2x)} + \frac{1}{(2L)} + S(x-d)$

$$S(L) \approx \frac{\ln\left(\dfrac{L}{x}\right)}{d} + \frac{1}{(2x)} + \frac{1}{(2L)} + S(x-d) \tag{Final}$$

3.2. Formula Analysis

3.2.1. Components of Formula

Variable 'x' . Variable 'x' in the formula, can be substituted with any term of the corresponding AP such that

1. $[x \in AP]$
2. $[a \leq x \leq L]$

Term 'S(x-d)'. $S(x-d) = \frac{1}{(a)} + \frac{1}{(a+d)} + \frac{1}{(a+2d)} + \dots\dots\dots\dots + \frac{1}{(x-d)}$

The term S(x-d) has to be calculated manually. This adds the component of manual work done to the equation. As the value of 'x' increases, the manual work done to find the answer would also increase.

Variability of the Formula. Due to the inclusion of 'x' and 'S(x-d)' the formula becomes variable in terms of complexity and accuracy.

As the value of 'x' increases, the formula becomes more accurate but also becomes more complex to calculate because of the term 'S(x-d)'.

We can conclude that Accuracy of the formula is directly proportional to the complexity of the formula. User can use this fact and change the accuracy or complexity according to his/her needs.

3.2.2. Formula Verification

Fundamentals . Sum of HP = S(L) = $\frac{1}{(a)} + \frac{1}{(a+d)} + \frac{1}{(a+2d)} + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + \frac{1}{(L)}$

Approximation = f(L) = $\frac{\ln\left(\frac{L}{x}\right)}{d} + \frac{1}{(2x)} + \frac{1}{(2L)} + S(x-d)$

Error in Formula = $\Delta E = S(L) - f(L)$

Accuracy in percentage = $A\% = \left[\frac{f(L)}{S(L)}\right] \times 100$

Percent of terms calculated manually = $W\% = \frac{(x-a)}{(L-a)} \times 100$

Let's Consider the HP. $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots\ldots\ldots\ldots\ldots, \frac{1}{100}$ $\quad a = 1,\ d = 1,\ n = 100,\ L = 100$

$S(100) = 5.187377518$ (*calculated manually*)

For x=a=1

$$f(100) = \frac{\ln\left(\frac{100}{1}\right)}{1} + \frac{1}{2\times1} + \frac{1}{2\times100} + S(0)$$
$$= \ln(100) + \frac{1}{2} + \frac{1}{200}$$
$$= 5.110170186$$

$\triangle E = 0.0772$

$A\% = 98.512\ \%$

$W\% = 0\%$

For x=(a+5d)=6

$$f(100) = \frac{\ln\left(\frac{100}{6}\right)}{1} + \frac{1}{2\times6} + \frac{1}{2\times100} + S(5)$$
$$= \ln\left(\frac{50}{3}\right) + \frac{1}{12} + \frac{1}{200} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$
$$= 5.185077383$$

$\triangle E = 0.0023$

$A\% = 99.956\ \%$

$W\% = 5\ \%$

For x=(a+20d)=21 $f(100) = \frac{\ln\left(\frac{100}{21}\right)}{1} + \frac{1}{2\times21} + \frac{1}{2\times100} + S(20)$

$= \ln\left(\frac{100}{21}\right) + \frac{1}{42} + \frac{1}{200} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{20}$

$= 5.187196929$

$\triangle E = 0.00018$

$A\% = 99.996\ \%$

$W\% = 20\%$

Let's Consider the HP. $\frac{1}{3}, \frac{1}{7}, \frac{1}{11}, \frac{1}{15}, ..., \frac{1}{99}$ $a = 3$, $d = 4$, $n = 25$, $L = 99$

$S(99) = 1.078688093\ (calculated\ manually)$

For x=a=3 $f(99) = \frac{\ln\left(\frac{99}{3}\right)}{4} + \frac{1}{2\times3} + \frac{1}{2\times99} + S(0)$

$= \frac{\ln(33)}{4} + \frac{1}{6} + \frac{1}{198}$

$= 1.045844062$

$\triangle E = 0.03284403$

$A\% = 96.928\ \%$

$W\% = 0\ \%$

For x=(a+5d)=23 $f(99) = \frac{\ln\left(\frac{99}{23}\right)}{4} + \frac{1}{2\times23} + \frac{1}{2\times99} + S(19)$

$= \frac{\ln\left(\frac{99}{23}\right)}{4} + \frac{1}{46} + \frac{1}{198} + \frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \frac{1}{19}$

$= 1.078093857$

$\triangle E = 0.0005942$

$A\% = 99.945\ \%$

$W\% = 20\%$

For x=(a+10d)=43 $f(99) = \frac{\ln\left(\frac{99}{43}\right)}{4} + \frac{1}{2\times43} + \frac{1}{2\times99} + S(39)$

$= \frac{\ln\left(\frac{99}{43}\right)}{4} + \frac{1}{86} + \frac{1}{198} + \frac{1}{3} + \frac{1}{7} + \frac{1}{11} + ... + \frac{1}{39}$

$= 1.078541975$

$\triangle E = 0.000146118$

$A\% = 99.986\ \%$

$W\% = 40\%$

4. Conclusion

1. The Formula provides us with an answer to Sum of Harmonic Progression. This formula is the closest approximation of HP till date
2. For an HP with low number of terms, the sum can be calculated manually. But when the number of terms become huge, the process of manual calculation becomes tedious and time-consuming. The formula devised in this paper eliminates the aforementioned problems without increasing the complexity of the formula.
3. Accuracy and manual work done to find the answer by using the formula are both variable. Accuracy of the formula is directly proportional to manual work done for calculation.
4. When sum of HP is the principal result in a project, then a low value of 'x' can be taken, as the error of the formula will not affect anything other than the approximation itself. But if the sum of HP is a sub-ordinate result, then a high value of 'x' should be taken, as the higher accuracy would decrease the chances of deviation in the final result.

4.1. Future Research and Application

1. The resultant formula can be used to find the approximation of sum of Harmonic series including the infinite series.
2. One can also use the Euler's constant to determine the errors in the approximation.
3. Sum of harmonic progression might also have applications in the approximation theory.
4. Further use of Analysis could be used to find a better approximation.

5. List of abbreviations

HP :- Harmonic Progression
AP :- Arithmetic Progression
S(L) :- Sum of Harmonic Progression
f(L) :- Approximation of Harmonic Progression
Ar(Figure 1) :- Area of Figure 1

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