Approximation of Sum of Harmonic Progression

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Abstract

Sum of Harmonic Progression is an old problem. While a few complex approximations have surfaced, a simple and efficient formula hasn't. Motive of the paper is to find a general formula for sum of harmonic progression without using 'summation' as a tool. This is an approximation for sum of Harmonic Progression for numerical terms. The formula was obtained by equating the areas of graphs of Harmonic Progression and curve of equation ($y=x^{-1}$). Formula also has a variability that makes it more suitable for different users with different priorities in terms of accuracy and complexity.

Introduction

- Sum of Harmonic Progression

Harmonic Progression (HP) is a series of numbers where each term is the reciprocal of the corresponding term of an Arithmetic Progression (AP).

Consider an AP, where $(a = first\ term)$. $(d = common\ difference)$. $(n = number\ of\ terms)$. $(L = last\ term = a + (n-1)d.)$

The AP= $(a), (a + d), (a + 2d), (a + 3d), \dots \dots \dots \dots \dots, (L)$

For this AP, the corresponding HP =

$$\frac{1}{(a)}, \frac{1}{(a+d)}, \frac{1}{(a+2d)}, \frac{1}{(a+3d)}, \dots \dots \dots \dots \dots .., \frac{1}{(L)}$$

Sum of this HP =

$$\frac{1}{(a)} + \frac{1}{(a+d)} + \frac{1}{(a+2d)} + \frac{1}{(a+3d)} + \dots \dots \dots \dots \dots \dots + \frac{1}{(L)}$$

- Why Sum of HP is incalculable

Even though n^th term of HP is determinable, its value cannot be calculated only using the first term. This is due to a lack of specific or similar arithmetic or geometric relation between consecutive terms of HP. Because of which, arithmetic value of n^th term of HP cannot be calculated using a general equation.

E.g. If we were to use the sum of Geometric Progression as a tool, we would have to add, sums of geometric progressions which include each and every term, including the prime numbers. Since Prime numbers have no other divisor other than 1, number of Geometric Progressions to be considered becomes huge, which makes it impossible to calculate the Sum of HP with a general Formula devised using the sum of Geometric Progression.

Formula for Approximation of Sum of HP

Sum of Harmonic Progression = *S(L)*

$$S(L) = \frac{1}{(a)} + \frac{1}{(a+d)} + \frac{1}{(a+2d)} + \frac{1}{(a+3d)} + \dots\dots\dots\dots\dots + \frac{1}{(L)}.$$

Approximation of Sum of HP = *f(L) ≈ S(L)*

$$f(L) = \frac{\ln[L/x]}{d} + \frac{1}{2L} + \frac{1}{2x} + S(x-d)$$

Where, $(L = last\ term\ of\ the\ corresponding\ AP.)$
$(d = common\ difference)$. $(x = any\ term\ of\ the\ corresponding\ AP.)$
$\left(S(x-d) = \frac{1}{(a)} + \frac{1}{(a+d)} + \frac{1}{(a+2d)} + \cdots \dots\dots\dots\dots. + \frac{1}{(x-d)}\right)$
(Term 'S(x-d)' has to be calculated manually)

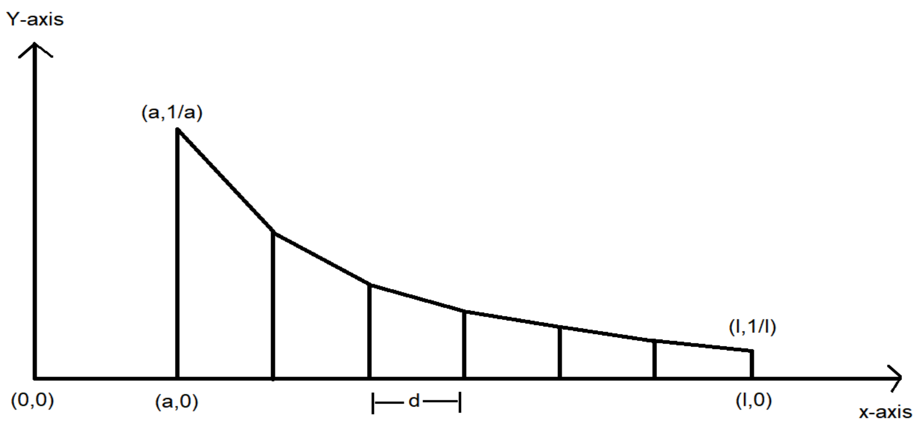
Formula was obtained by equating the area under the curve of equation, $[y = \left(\frac{1}{X}\right)]$ with the area enclosed in the graph of Harmonic Progression.

The variable ‘x’ adds variability to the accuracy of the formula. The formula changes according to the user’s need. But due to the addition of term [S(x-d)] which has to be calculated manually, the work done also becomes variable.

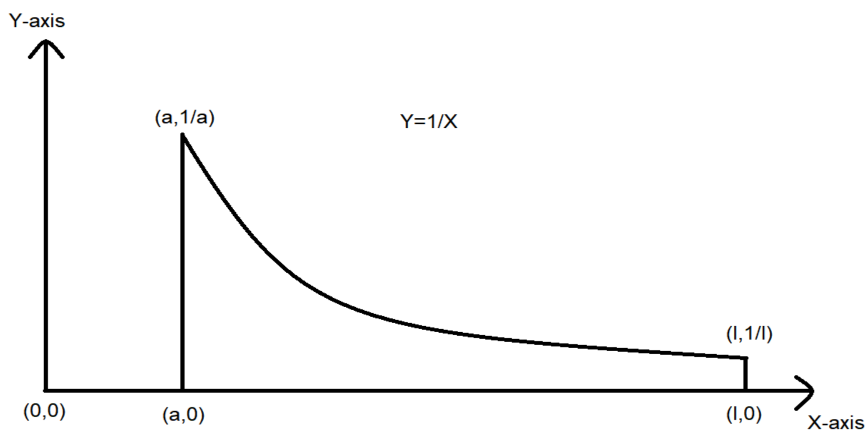
Proof

- Fundamentals for the proof.

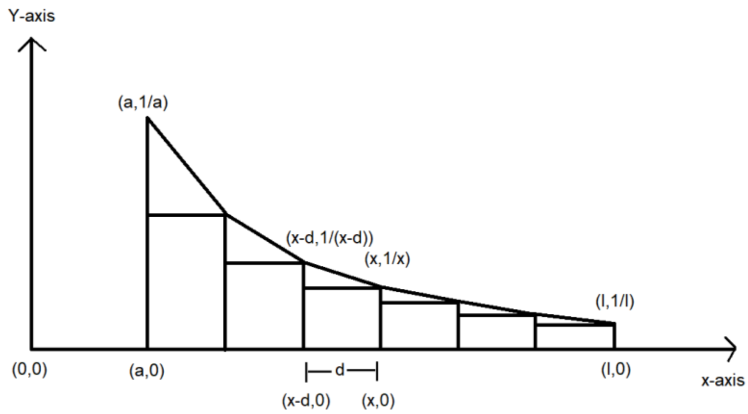
① Graph of Harmonic Progression.



② Graph of curve of equation [y=1/x].



- Area enclosed in ①



The area enclosed in is formed of sum of areas of all Triangles and Rectangles.

- $Area\ of\ one\ Triangle\ =\ ½\ \times\ base\ \times\ height$

  $Base\ for\ any\ triangle\ =\ d.$

  $Height\ for\ a\ triangle = [\frac{1}{(x-d)} - \frac{1}{(x)}],\ [x \in AP]$

  $Area\ of\ one\ triangle\ =\ ½\ \times\ d\ \times [\frac{1}{(x-d)} - \frac{1}{(x)}]$

  $Sum\ of\ Areas\ of\ All\ Triangles =$

$$\sum_{x=(a+d)}^{L} \frac{d}{2}[\frac{1}{(x-d)} - \frac{1}{(x)}],\ [x \in AP]$$

$$= \frac{d}{2} \times \left[\frac{1}{(a)} - \frac{1}{(a+d)} + \frac{1}{(a+d)} - \cdots \ldots \ldots \ldots \ldots \ldots . - \frac{1}{(L)}\right]$$

$$= \frac{d}{2} \times \left[\frac{1}{(a)} - \frac{1}{(L)}\right] \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots . \ ❶$$

- *Area of one Rectangle = base × height*

  Base for any rectangle = d.

  Height of a rectangle = $\frac{1}{(x)}$ *where x is any term of the AP.*

  Area of one rectangle = $d\ \times \frac{1}{(x)}.$

Sum of Areas of all rectangles =

$$\sum_{x=(a+d)}^{L} d \times \frac{1}{(x)}, [x \in AP]$$

$$= d \times [\frac{1}{(a+d)} + \frac{1}{(a+2d)} + \frac{1}{(a+3d)} + \cdots \ldots \ldots \ldots \ldots . + \frac{1}{(L)}]$$

$$= d \times [S(L) - \frac{1}{(a)}] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{❷}.$$

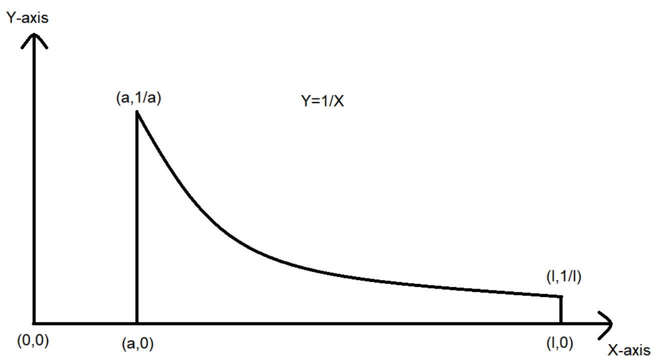
- ➢ Area enclosed in ① = ❶ + ❷.

$$Ar(①) = \frac{d}{2} \times \left[\frac{1}{(a)} - \frac{1}{(L)}\right] + d \times \left[S(L) - \frac{1}{(a)}\right].$$

$$= d \times \left\{\left[S(L) - \frac{1}{(a)}\right] + \left[\frac{1}{(2a)} - \frac{1}{(2L)}\right]\right\}$$

$$= d \times \left[S(L) - \frac{1}{(2a)} - \frac{1}{(2L)}\right] \ldots\ldots\ldots\ldots\ldots\ldots \text{❸}.$$

- Area enclosed in ②



- ➢ Area enclosed in ② is equal to the area under the curve of equation [y=1/x].

$$= \int_a^L y dx = \int_a^L \frac{1}{(x)} dx$$

$= \ln[x]$ with the *(upper limit = L)* and *(lower limit = a)*

$$= \ln[L/a] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{❹}.$$

- Equating ① and ②.

We implement the logic, that area enclosed in the graph of Harmonic Progression (①) is approximately equal to the area under the curve of equation [Y=(1/x)] (②).

Therefore,

Ar(①) ≈ Ar(②).

Which would mean that ❸ ≈ ❹.

Therefore,

$$d \times \left[S(L) - \frac{1}{(2a)} - \frac{1}{(2L)}\right] \approx \ln[L/a]$$

$$= S(L) - \frac{1}{(2a)} - \frac{1}{(2L)} \approx \frac{\ln[L/a]}{d}$$

$$= S(L) \approx \frac{\ln[L/a]}{d} + \frac{1}{(2a)} + \frac{1}{(2L)}$$

- Addition of variable *'x'*

With the increase in the value of 'a', the difference of slopes between consecutive terms decreases. Due to this the error i.e. S(L) - f(L) also decreases. So if the first few terms of an HP are reduced or calculated manually, the error of the formula will also be reduced. So if 'a' were to be substituted with 'x', for any term of AP, error will decrease.

For a sum of HP

$$= \frac{1}{(a)} + \frac{1}{(a+d)} + \frac{1}{(a+2d)} + \dots\dots\dots\dots\dots\dots + \frac{1}{(L)}$$

We divide this sum into two parts. i.e.

1st part $= \frac{1}{(a)} + \frac{1}{(a+d)} + \frac{1}{(a+2d)} + \dots\dots\dots.\dots.. + \frac{1}{(x-d)}$

2nd part $= \frac{1}{(x)} + \frac{1}{(x+d)} + \frac{1}{(x+2d)} + \dots\dots\dots\dots\dots.. + \frac{1}{(L)}$

The value of 2nd part is calculated using the above formula.

2nd part $= \frac{\ln[L/x]}{d} + \frac{1}{(2x)} + \frac{1}{(2L)}$

The value of 1st Part = *S(x-d)*, is calculated manually, and then added to both sides.

- $\boldsymbol{S(L) \approx \frac{\ln[L/x]}{d} + \frac{1}{(2x)} + \frac{1}{(2L)} + S(x-d)}$

Formula Analysis

- ‘x’ component of the formula.

  Variable ‘x’ in the formula, can be substituted with any term of the corresponding AP.Such that

  1. $[x \ \in AP]]$
  2. $[a \leq x \leq L]$

- Variability of the formula.

  Due to the inclusion of variable ‘x’, the accuracy of the formula is also variable.

  But as the term ‘S(x-d)’ is to be calculated manually, the work done (manual) also varies.

  When the value of ‘x’ is taken as the nth term of the AP, (n-1) terms will have to be calculated manually.

  When the value of ‘x’ increases, the accuracy of the formula also increases. But the number of terms to be calculated also increases and so the manual work done also increases.

  This way, one can change the formula according to one’s needs.

  For an HP with ‘n’ terms, the number of approximations is also equal to ‘n’. User can select the one that he/she finds suitable.

- Formula Verification.

$$S(L) = Sum\ of\ HP = \frac{1}{(a)} + \frac{1}{(a+d)} + \frac{1}{(a+2d)} + \cdots \ldots \ldots \ldots + \frac{1}{(L)}$$

$$f(L) = Approximation = \frac{\ln\left[\frac{L}{x}\right]}{d} + \frac{1}{2x} + \frac{1}{2L} + S(x-d)$$

$$\Delta E = error\ in\ formula = S(L) - f(L).$$

$$A = Accuracy\ in\ percentage = \left[\frac{f(L)}{S(L)}\right] \times 100$$

$$W = number\ of\ terms\ calculated\ manually = \frac{(x-a)}{d}$$

Let’s consider the HP

$$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \ldots \ldots \ldots \ldots \ldots \ldots \frac{1}{100}.$$

a=1, d=1, n=100, L=100.

S(100) = 5.187377518 (calculated manually).

- For x=a=1.

$$f(100) = \frac{\ln[100]}{1} + \frac{1}{2} + \frac{1}{200} + S(0)$$

= 5.110170186.

(ΔE = 0.0772) ; (A = 98.512 %) ; (W=0).

- For x=(a+5d)= 6

$$f(100) = \frac{\ln\left[\frac{100}{6}\right]}{1} + \frac{1}{12} + \frac{1}{200} + S(5)$$

$$= \ln\left[\frac{50}{3}\right] + \frac{1}{12} + \frac{1}{200} + 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$$

= 5.185077383.

(ΔE = 0.0023) ; (A = 99.956 %) ; (W = 5).

- For x= a+20d = 21

$$f(100) = \frac{\ln\left[\frac{100}{21}\right]}{1} + \frac{1}{42} + \frac{1}{200} + S(20)$$

$$= \ln\left[\frac{100}{21}\right] + \frac{1}{42} + \frac{1}{200} + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots \dots \dots \dots . + \frac{1}{20}$$

= 5.187196929.

(ΔE = 0.00018) ; (A = 99.996 %) ; (W = 20).

Let’s consider the HP

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \dots \dots \dots \dots \dots + \frac{1}{99}.$$

S(99) = 1.078688093. (Calculated manually)

a=3, d=4, n=25, L=99

- x = a= 3

$$f(99) = \frac{\ln\left[\frac{99}{3}\right]}{4} + \frac{1}{6} + \frac{1}{198} + S(0)$$

$$= 1.045844062.$$

$(\Delta E = 0.03284403)$, $(A = 96.928\%)$, $(W = 0)$

- x = (a+3d) = 15

$$f(99) = \frac{\ln\left[\frac{99}{15}\right]}{4} + \frac{1}{30} + \frac{1}{198} + \frac{1}{3} + \frac{1}{7} + \frac{1}{11}$$

$= 1.077250818.$

$(\Delta E = 0.001437275),\quad (A = 99.866\%),\quad (W = 3)$

- x = (a+7d) = 35

$$f(99) = \frac{\ln\left[\frac{99}{35}\right]}{4} + \frac{1}{70} + \frac{1}{198} + \frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \frac{1}{15} + \ \dots\dots\dots. + \frac{1}{31}$$

$= 1.078450342$

$(\Delta E = 0.00023775),\quad (A = 99.978\%),\quad (W = 7)$

Sum of Infinite Harmonic Series

For two infinite harmonic series with opposite signs, the logs of infinity get cancelled.

For ex. Considering an Harmonic Series

$$S_2(\infty) = \frac{1}{(a)} - \frac{1}{(a+d)} + \frac{1}{(a+2d)} - \frac{1}{(a+3d)} + \cdots \dots\dots. + \frac{1}{\infty} - \frac{1}{\infty}$$

$$= \frac{1}{(a)} + \frac{1}{(a+2d)} + \cdots \dots\dots. + \frac{1}{\infty} - \left[\frac{1}{(a+d)} + \frac{1}{(a+3d)} + \cdots \dots\dots + \frac{1}{\infty}\right]$$

$$= \frac{\ln[{}^{\infty}/_{a}]}{2d} + \frac{1}{2a} + \frac{1}{2\infty} - \left[\frac{\ln\left[{}^{\infty}/_{(a+d)}\right]}{2d} + \frac{1}{2(a+d)} + \frac{1}{2\infty}\right]$$

$$= \frac{\ln\left[\frac{\infty}{a} \times \frac{(a+d)}{\infty}\right]}{2d} + \frac{d}{2a(a+d)}$$

$$= \frac{\ln\left[\frac{(a+d)}{a}\right]}{2d} + \frac{d}{2a(a+d)}$$

$$S_2(\infty) > \frac{\ln\left[\frac{(a+d)}{a}\right]}{2d} + \frac{d}{2a(a+d)}$$

Ex.

$$S(\infty) = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots \ldots \ldots \ldots + \frac{1}{\infty} - \frac{1}{\infty}$$

$$S(\infty) > \frac{\ln\left[\frac{2}{1}\right]}{2 \times 1} + \frac{1}{2 \times 1 \times 2}$$

$$S(\infty) > \frac{\ln[2]}{2} + \frac{1}{4}$$

- Conclusion

The Formula provides us with an answer to an old problem of sum of harmonic progression.
For an HP with low number of terms, the sum can be calculated manually. But when the number is high, the process of manual calculation becomes tedious and time-consuming. The formula devised in this paper eliminates the aforementioned problems without increasing the complexity of the formula.
Accuracy of the approximation can be increased with a little effort, in the form of manual calculation.
The approximation is applicable for any HP with numerical terms and is devoid of 'summation' as a tool. The difference in the error of the approximation of consecutive number of terms decreases exponentially with increase in the number of terms.
The term S(x-d) is to be calculated on a calculator or by vulgar fraction addition (manual) whichever seems easier. Due to this, the manual work done to find the answer becomes variable.
When Sum of HP, is the principal result, a low value of 'x' can be taken, as the error in the formula will not affect anything other than the approximation itself. But if the approximation is a sub-ordinate result and is needed to find further results, a high value of 'x' should be taken, so that error in the approximation does not affect other results.

- Future Research and Potential.

Further Research can be done to find the Sum of Harmonic Progression. Since this is an approximation and is only 100% accurate, in 1 of the 'n' number of times, further research might give us a definite result.
Sum of HP, can be used as a tool, for other researches, in various other fields. It is more like a tool for other results rather than it being the principal result. With this formula, it will be easier and more efficient, pathway to problems related to sum of HP.

This research might also have significant result into the music theory, based on HP, and chord wavelengths.

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- References:-

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