The Reformulation of Faraday's Law Using Electric Displacement Instead of Electric Field

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***Abstract*—In 1864, James Clerk Maxwell formulated a set of electromagnetic equations to describe the interactions between electric and magnetic fields, now known as Maxwell's equations in his honor. In this paper, it will be shown using a thought experiment, Faraday's law formulated using electric displacement, instead of electric field, is a better formulation, consistent with the definitions of electric field and electric displacement. Maxwell's equations will be reformulated to satisfy the constituent relations, reflecting the change in the formulation of Faraday's law.**

***Index Terms*—Maxwell's equations, electric displacement, electric field, magnetic field, electric-flux density, magnetic-flux density, Faraday's law, Ampere's law**

I. Introduction

JAMES Clerk Maxwell published 3 famous papers on electromagnetism [1] – [3]. He translated Faraday's experiment results into a mathematical equation, known as Faraday's law today, and introduced the displacement current. In his famous 1864 paper [3], he presented a comprehensive theory of the electromagnetic field, including his hypothesis of the existence of electromagnetic waves, which was confirmed by Heinrich Hertz in 1893 [4]. The motivation to reformulate Faraday's law will be discussed in detail first, followed by the reformulation of Maxwell's equations.

This paper requires a good understanding of the meaning of electric field and electric displacement, or electric-flux density, as its also known. A new perspective on their definitions will be presented in Section II. The reader is referred to Ref. [5] for more details.

In this paper, Maxwell's equations will be presented in electrostatic units (ESU), and will be summarized in Section III. The reason for choosing this particular system of units will be explained in Section III. The conclusions and the formulations presented in this paper, however, are not limited to ESU, but applicable to any system of equations, including SI. The differences are only in the mathematical formulations, and not the physics.

The motivation to reformulate Faraday's law will be explained by a thought experiment in Section IV, which is simple, and easy to understand. How electric field and electric displacement are viewed in the current formulation will be presented in Section V, which however, do not agree with their definitions.

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A reformulation of Faraday's law requires changes to the remaining Maxwell's equations, so that the constitutive relations in Maxwell's equations are satisfied, to be able to derive the wave equation, for example. This will be discussed in Sections VI-VIII. The derivation of the wave equation in the reformulated equations will be presented in Section IX, followed by the set of equations in electrostatics and magnetostatics in Section X – Section XI.

Other possibilities for the reformulation of Ampere's law, to reflect the change in Faraday's law, will be considered in Section XII. However, it will be shown that these are incorrect.

II. Electric Field $\vec{E}$ vs Electric Displacement $\vec{D}$

The meaning of electric field and electric displacement will be derived from Gauss's law and Faraday's experiment with spherical capacitors in this section. The reader is referred to Ref. [5] for more details.

In ESU, Coulomb's law is used to define the unit electric charge from the equation,

$$\vec{F} = \frac{q_1\, q_2}{r^2}\,\hat{r}, \tag{1}$$

where $\vec{F}$ is the force between charges q_1 and q_2, separated by distance r, and $\hat{r}$ is the unit vector in the direction of the force. The electric field $\vec{E}$ at any point is defined as the force per unit charge,

$$\vec{E} = \frac{\vec{F}}{q}, \tag{2}$$

where $\vec{F}$ is the force experienced by charge q. The above definition of Coulomb's law will be limited to charges in free space. Using the above two equations, Gauss's law for charges in free space in the electrostatic case, can be derived as

$$\oint_S \vec{E}\cdot d\vec{A} = 4\pi q_{enc}, \tag{3}$$

where S is a closed 3D surface enclosing charge q_{enc}. The derivation of the above equation is presented in Ref. [5], and is not repeated here.

Faraday's experimental setup is shown in Fig. 1. Faraday used two identical capacitors, A and B, with the two inner metal spheres, p and p', connected together by a wire, as well as the two outer metal spheres q and q'. A metal is an equipotential volume, and so are metal objects connected together. Therefore, p and p' are at the same potential, and so are q and q'. Since the inner spheres are connected, as well as the outer spheres, the potential difference between the inner sphere and the outer sphere in A, V_{pq}, is equal to that of B, $V_{p'q'}$,

$$V_{pq} = V_{p'q'}. \tag{4}$$

Since the capacitors are identical, and voltage is the path integral of the electric field, the electric fields in the cavities of A and B, $\vec{E}_A$ and $\vec{E}_B$, are equal,

$$\vec{E}_A = \vec{E}_B, \tag{5}$$

at the same point relative to the respective centers of A and B, in each of the cavities. This is true, independent of the dielectric material that fills the cavities of A and B.

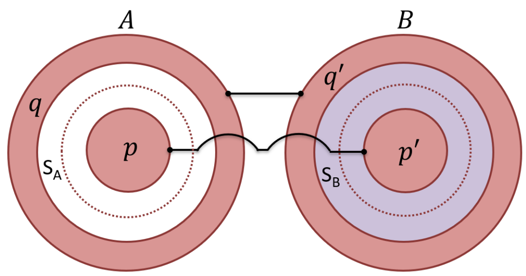

Fig. 1. The experimental setup to study the variation of the charge stored in a capacitor with different dielectric materials.

In the cavity of A, the material is always kept as air, but the material in B is varied. Capacitor A with the unfilled cavity acts as a reference for the experiment. The capacitors are charged simultaneously by connecting the inner and the outer spheres to the terminals of a Wimshurst machine. A detailed explanation of how a Wimshurst machine works is presented in Ref. [5].

Faraday studied the ratio of the charge stored in A, Q_A, and B, Q_B. The quantity of charge stored in a capacitor can be measured using a ballistic galvanometer. The detailed methodology is presented in Ref. [5]. If the dielectric material in both A and B is air, by symmetry, the charge stored in the two identical capacitors are equal. The ratio of the charge stored $Q_B : Q_A$ is 1. However, in the case when B is filled with a dielectric material other than air, Faraday observed that the ratio is greater than 1. This ratio has a special name and is called the relative permittivity of the dielectric material, denoted by the symbol ϵ_r, where r stands for *relative*, and it means the permittivity of a material relative to air. To be precise, the cavity in Capacitor A must be vacuum, which is the absence of any material, including air. The reference dielectric material of air will be assumed for simplicity.

Faraday observed that Capacitor B, whose cavity is filled with a dielectric material, stores ϵ_r times more charge than Capacitor A, whose cavity is unfilled, or contains air. If $\vec{E}_A$ is the electric field in capacitor A, applying Gauss's law in free space, in Equation 3,

$$\oint_{S_A} \vec{E}_A \cdot d\vec{A} = 4\pi\, Q_A, \tag{6}$$

where $\pm Q_A$ is the charge stored in the spheres of A, which reside on the outside of the inner sphere, and the inside of the outer spherical shell [5], and S_A is the spherical Gaussian surface in the cavity, as shown by the dotted line. Solving the above equation, the electric-field strength in the cavity is $\propto Q$.

If this same equation is applied to Capacitor B, the field strength in the cavity is ϵ_r times greater, since ϵ_r times more charge is stored in Capacitor B. By definition, voltage is the path integral of the electric field, and therefore, $V_{p'q'}$ is ϵ_r times greater than V_{pq}. Since $V_{pq} = V_{p'q'}$, this contradicts the experiment results. This can be resolved with the explanation presented next.

If the dielectric material in the cavity of Capacitor B reduces the electric-field strength by ϵ_r, then the electric fields in the cavities of A and B are equal. If the electric fields are equal, this means that the voltage $V_{pq} = V_{p'q'}$. This explains the reason that more charge is present in Capacitor B: the additional charge is present to overcome the reduction in the electric field caused by the dielectric material, so that the electric fields in the cavities of A and B are equal. This explanation can be captured as

$$\vec{E}_{material} = \frac{\vec{E}_{air}}{\epsilon_r}, \tag{7}$$

where $\vec{E}_{material}$ is the electric field at any point in a material with permittivity ϵ_r, and $\vec{E}_{air}$ is the electric field in air, or the electric field that would have existed at that point, if the material does not reduce the field. This observation will be used to formulate Gauss's law in the remainder of this section, taking into account the reduction of electric-field strength in a dielectric material.

The ratio of the charges stored in A and B is the relative permittivity ϵ_r,

$$Q_A = \frac{Q_B}{\epsilon_r}. \tag{8}$$

Since the potential difference between the outer and the inner conductors are the same in both the identical capacitors, as noted in Eq. 4,

$$\oint_{S_B} \vec{E}_B \cdot d\vec{A} = \oint_{S_A} \vec{E}_A \cdot d\vec{A}, \tag{9}$$

where S_B is a spherical Gaussian surface lying in the cavity of B, and of the same radius as S_A, shown by the dotted circles in Figure 1. From the above equations,

$$\oint_{S_B} \vec{E}_B \cdot d\vec{A} = 4\pi \frac{Q_B}{\epsilon_r}. \tag{10}$$

Rearranging the above equation,

$$\oint_{S_B} \epsilon_r \vec{E}_B \cdot d\vec{A} = 4\pi\, Q_B, \tag{11}$$

is the general form of Gauss's law.

In this example, the Gaussian surface is present in an uniform dielectric material. ϵ_r is moved inside the integral, which will account for the variation in the dielectric material over S. Gauss's law is also valid in this case. However, the validity of Gauss's law in any type of material medium, uniform or non-uniform, isotropic or anisotropic, linear or non-linear, or in the case of time-varying fields, can be proven with the current-continuity equation, and is presented in Ref. [5] (See also Ref. [6]- [7]). In other words, Gauss's law is always valid!

From the above equations, the general form of Gauss's law is written as

$$\oint_S \vec{D} \cdot d\vec{A} = 4\pi\, q_{enc}, \tag{12}$$

where S is the Gaussian surface enclosing charge q_{enc}.

The integrand in the above equation is assigned a new vector-field quantity $\vec{D}$, and is called electric displacement,

$$\vec{D} = \epsilon_r \vec{E}. \tag{13}$$

$\vec{D}$ and $\vec{E}$ are related at any point by the above equation.

$\vec{D}$ is typically viewed as a vector field that satisfies Equation 12, or as a mathematical relation in Equation 13. However, a new meaning emerges, when Equation 7 is substituted in Equation 13, resulting in

$$\vec{D} = \vec{E}_{air}. \tag{14}$$

The above equation states that electric displacement is the same as the electric field in air, or in other words, electric displacement at any point is the same as the electric field that is not "modified", "altered", or reduced in strength by the dielectric material at that point.

In ESU, electric field $\vec{E}$ and electric displacement $\vec{D}$ have the same units, and from dimensional analysis applied to the above equations [5] [8],

$$[\vec{E}] = [\vec{D}] = \frac{gm^{1/2}}{cm^{1/2} \cdot sec}. \tag{15}$$

This is a useful property for the new set of equations formulated, as explained in Section IV, since $\vec{E}$ can be replaced by $\vec{D}$, without requiring modifications to any of the units of any electrical quantity. Although the new set of equations will be formulated in ESU, it can also be derived in other systems of units, including the present SI system.

Note that since $\vec{D}$ is also an electric field in ESU, the force $\vec{F}$ on an electric charge q

$$\vec{F} = q\vec{E}, \tag{16}$$

can also be applied to $\vec{D}$,

$$\vec{F} = q\vec{D}. \tag{17}$$

The difference between electric field $\vec{E}$ and electric displacement $\vec{D}$ is that $\vec{E}$ at any point, is the electric field that *exists* in a material, and $\vec{D}$ is the electric field that *would have* existed at that point, if the material at that point does not modify, or reduce the electric field $\vec{D}$.

Although the above two equations are specific to ESU, it is possible to formulate Maxwell's equations in SI units, such that $\vec{E}$ and $\vec{D}$ have the same units. However, the constant ϵ_o will have to be included in other equations.

III. Maxwell's Equations in Electrostatic Units

Maxwell's equations in the differential form in ESU [5], is summarized in this section. Gauss's law in ESU satisfies the relation,

$$\nabla \cdot \vec{D} = 4\pi\rho, \tag{18}$$

where ρ is volume-charge density, $\vec{D}$ is the electric displacement. Ampere's law is

$$\nabla \times \vec{H} = 4\pi\vec{J} + \frac{\partial \vec{D}}{\partial t}, \tag{19}$$

where $\vec{J}$ is conduction current, which is the current resulting from the flow of electric charges, and $\vec{H}$ is the magnetic field. Faraday's law is

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{20}$$

where $\vec{E}$ is the electric field generated by a time-varying magnetic-flux density $\vec{B}$. The divergence-free condition of $\vec{B}$ is

$$\nabla \cdot \vec{B} = 0. \tag{21}$$

In ESU, the relation between magnetic-flux density $\vec{B}$ and magnetic field $\vec{H}$ is

$$\vec{B} = \frac{\mu_r}{c^2}\vec{H}, \tag{22}$$

where c is the speed of light, and μ_r is the relative permeability. The relation between $\vec{D}$ and $\vec{E}$ is, repeating Equation 13,

$$\vec{D} = \epsilon_r \vec{E}. \tag{23}$$

IV. Reformulation of Faraday's Law

The motivation to reformulate Faraday's law using electric displacement, instead of electric field, will be presented using a very simple thought experiment.

Two cases are analyzed: the material in Case 1 in Figure 2 is a uniform dielectric material of relative permittivity ϵ_{r1}, while the region in Case 2 is filled with a material of permittivity ϵ_{r2}. The time-varying magnetic-flux density $\vec{B}$ are equal in both the cases. Case 1 and Case 2 are identical, except for the permittivity of the material that fills the space. Since $\vec{B}$ are equal in both the cases, by Faraday's law,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{24}$$

the electric field $\vec{E}$ generated in each of the cases must be equal as well.

The electric displacement, however, $\vec{D}_1$ in Case 1, and $\vec{D}_2$ in Case 2, are different,

$$\vec{D}_1 = \epsilon_{r1}\vec{E} \tag{25}$$

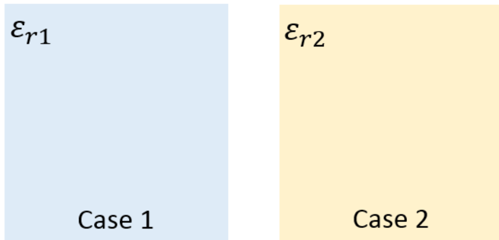

Fig. 2. Two identical imaginary loops in a uniform medium with relative permittivity ϵ_{r1} and ϵ_{r2}.

$$\vec{D}_2 = \epsilon_{r2}\vec{E} \neq \vec{D}_1, \tag{26}$$

since the material properties are different.

The generation of $\{\vec{D},\ \vec{E}\}$ by a time-varying $\vec{B}$ can be viewed as a two-step process: in Step 1, by definition, electric displacement $\vec{D}$ is the field unmodified by a material, as seen earlier, is the field generated. In the second step, the material modifies the field, resulting in the electric field $\vec{E}$. Since $\vec{B}$ is the same in both the cases, $\vec{D}_1$ must be equal to $\vec{D}_2$. The permittivity of the materials are different in the two cases, and therefore, the electric fields $\vec{E}_1$ and $\vec{E}_2$ must be different, since the different materials in the two cases, modify the electric displacement differently, as noted in Section II. In the current formulation of Faraday's law, however, this is the opposite, as noted in the above equations, where $\vec{E}_1$ and $\vec{E}_2$ are equal, while $\vec{D}$ is different in the two cases, when it should be the other way around.

This is the motivation to reformulate Faraday's law as

$$\nabla \times \vec{D} = -\frac{\partial \vec{B}}{\partial t}, \tag{27}$$

written using electric displacement $\vec{D}$, instead of electric field $\vec{E}$. From the above equation, $\vec{D}$ generated in both the cases are equal, since time-varying $\vec{B}$ is the same in both the cases. The material modifies $\vec{D}$ differently. Using Equation 23, the electric field $\vec{E}_1$ and $\vec{E}_2$ can be calculated as

$$\vec{E}_1 = \frac{\vec{D}}{\epsilon_{r1}} \tag{28}$$

$$\vec{E}_2 = \frac{\vec{D}}{\epsilon_{r2}} \neq \vec{E}_1, \tag{29}$$

and the results are now consistent with the definitions of electric field and electric displacement, since the values of $\vec{E}$ are different in both the cases, but $\vec{D}$ is the same.

One may argue that both Faraday's law and Ampere's law are coupled, and both need to be taken into account in the analysis. However, each of the equations must be meaningful and correct by itself, without any coupling with other equations. Moreover, if it is assumed that $\vec{B}$ is linearly varying in time, there is no coupling between Faraday's law and Ampere's law, and the generated electric field is non time-varying and constant. The argument presented would still be valid. But as mentioned earlier in the paragraph, each of the equations in Maxwell's equations must be correct, when viewed independently.

V. Present Interpretation of Faraday's Law

In the present formulation of Maxwell's equations, the relation between $\vec{D}$ and $\vec{E}$, is similar to the relation between $\vec{B}$ and $\vec{H}$, the magnetic-flux density and the magnetic field, respectively.

The presence of $\vec{H}$ within a magnetic material, modifies the field to $\vec{B}$ [5]. Likewise, the existing formulation of Faraday's law views $\vec{D}$ as the electric field modified by a material, in the presence of an applied electric field $\vec{E}$. From Equation 25 and Equation 26, there is an applied electric field $\vec{E}$ generated from a changing magnetic flux in a dielectric material. The dielectric material then modifies the field to $\vec{D}_1$ and $\vec{D}_2$ that are different in Case 1 and Case 2.

However, this is inconsistent with the meaning of electric field and electric displacement, as noted in Section II. $\vec{E}$ is the electric field modified by a dielectric material, and $\vec{D}$ is the field that is not modified by the dielectric material.

VI. Additional Revisions

If Faraday's law is modified as Equation 27, this requires changes to other equations as well, so that the constiuent relations are satisfied. For example, Ampere's law must be modified as

$$\nabla \times \vec{H} = \vec{J} + \epsilon_r \frac{\partial \vec{D}}{\partial t}, \tag{30}$$

so that the wave equation can be derived. The derivation of the wave equation will be presented later in the paper. However, a serious problem that arises with the reformulation of Faraday's law, the violation of law of conservation of energy, needs to be fixed first, as explained next.

In the electrostatic case, there are no time-varying fields, and the right-hand side of Equation 27 reduces to 0,

$$\nabla \times \vec{D} = 0, \tag{31}$$

where $\vec{D}$ is a static field.

The conservative property of the electric field $\vec{E}$ must be met in electrostatics [5],

$$\nabla \times \vec{E} = 0, \tag{32}$$

and not necessarily $\vec{D}$, since $\vec{E}$ is the field that exists in a material, and not $\vec{D}$, as noted in Section II. Not meeting Equation 32 in electrostatics is a violation of the law of conservation of energy.

Additional revisions are needed, so that

$$\nabla \times \vec{E} = 0, \tag{33}$$

in the electrostatic case.

VII. Existing Formulation of Maxwell's Equations Written Using $\{\vec{E}_C, \vec{E}_F\}$ and $\{\vec{D}_C, \vec{D}_F\}$

The violation of law of conservation of energy will be fixed by first rewriting the existing formulation of Maxwell's equations, in a different manner. Electric displacement $\vec{D}$ will be split into $\vec{D}_C$ and $\vec{D}_F$, as well as electric field $\vec{E}$ into $\vec{E}_C$ and $\vec{E}_F$, explained next.

There are two sources of electric or electric displacement fields: the first is the Coulomb field from a charge or a charge distribution, which will be referred to as $\vec{D}_C$ and $\vec{E}_C$, and the second source is from a changing magnetic field, as written in Faraday's law, which will be referred to as $\vec{D}_F$ and $\vec{E}_F$. It will become clear later, why this field separation, can be used to fix the law of conservation of energy violation, explained in the previous section.

The magnetic vector potential $\vec{A}$ is defined as

$$\vec{B} = \nabla \times \vec{A}. \tag{34}$$

Substituting the above equation in the present formulation of Faraday's law,

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\left(\nabla \times \vec{A}\right). \tag{35}$$

Rearranging the above equation,

$$\nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0. \tag{36}$$

From calculus, if the curl of a vector field is 0, the vector field can be written as the gradient of a scalar field,

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla\Phi. \tag{37}$$

Rearranging the above equation,

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \tag{38}$$

$$= \vec{E}_C + \vec{E}_F. \tag{39}$$

From the above equation, $\vec{E}$ can be written as the sum of Coulomb field,

$$\vec{E}_C = -\nabla\Phi, \tag{40}$$

and the electric field generated from Faraday's law,

$$\vec{E}_F = -\frac{\partial \vec{A}}{\partial t}. \tag{41}$$

Applying the $\nabla\times$ operator on both sides of the above equation, and substituting Equation 34,

$$\nabla \times \vec{E}_F = -\frac{\partial \vec{B}}{\partial t}, \tag{42}$$

which is the same as Faraday's law. Intuitively, it can be observed that $\vec{E}_C$ is the electric field generated by electric charges. Since $\vec{E}_F$ is the electric field generated by Faraday's law, the other source of electric field $\vec{E}_C$, must be the electric field generated by electric charges. This can be proven mathematically, using Coulomb gauge,

$$\nabla \cdot \vec{A} = 0, \tag{43}$$

to show that $\vec{E}_C$ is the instantaneous Coulomb field [9] [10]. Since $\vec{E}_C$ is the gradient of a scalar potential in Equation 40, from calculus,

$$\nabla \times \vec{E}_C = 0. \tag{44}$$

Similar to Equation 39, $\vec{D}$ at any point can be written as

$$\vec{D} = \vec{D}_C + \vec{D}_F. \tag{45}$$

Using the above equation, Ampere's law is written as

$$\nabla \times \vec{H} = 4\pi\vec{J} + \frac{\partial}{\partial t}\left(\vec{D}_C + \vec{D}_F\right). \tag{46}$$

Applying the $\nabla\cdot$ operation on both the sides of the above equation,

$$\nabla \cdot \left(\nabla \times \vec{H}\right) = 4\pi\nabla \cdot \vec{J} + \frac{\partial}{\partial t}\nabla \cdot \left(\vec{D}_C + \vec{D}_F\right). \tag{47}$$

From calculus, the left-hand side reduces to 0. If

$$\nabla \cdot \left(\vec{D}_C + \vec{D}_F\right) = 4\pi\rho, \tag{48}$$

Equation 47 reduces to the current-continuity equation,

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}. \tag{49}$$

This shows that

$$\nabla \cdot \left(\vec{D}_C + \vec{D}_F\right) = 4\pi\rho, \tag{50}$$

or Gauss's law, must be satisfied for time-varying fields. The existing set of Maxwell's equations in ESU can be written as

$$\nabla \cdot \left(\vec{D}_C + \vec{D}_F\right) = 4\pi\rho \tag{51}$$

$$\nabla \cdot \vec{B} = 0 \tag{52}$$

$$\nabla \times \vec{E}_F = -\frac{\partial \vec{B}}{\partial t} \tag{53}$$

$$\nabla \times \vec{E}_C = \vec{0} \tag{54}$$

$$\nabla \times \vec{H} = 4\pi\vec{J} + \frac{\partial}{\partial t}\left(\vec{D}_C + \vec{D}_F\right), \tag{55}$$

and the above equations are also valid for time-varying fields [5]. In addition to the above equations,

$$\vec{D}_F = \epsilon_r \vec{E}_F \tag{56}$$

$$\vec{D}_C = \epsilon_r \vec{E}_C \tag{57}$$

$$\vec{B} = \mu_o \mu_r \vec{H} \tag{58}$$

$$\mu_o = \frac{1}{c^2}, c \approx 3.0 \times 10^{10}\, cm/s, \tag{59}$$

capture the effect of a material on the fields. The electric displacement and electric field at any point are

$$\vec{D} = \vec{D}_C + \vec{D}_F \tag{60}$$

$$\vec{E} = \vec{E}_C + \vec{E}_F. \tag{61}$$

VIII. Reformulation of Maxwell's Equations

Faraday's law is modified as

$$\nabla \times \vec{D}_F = -\frac{\partial \vec{B}}{\partial t}, \tag{62}$$

as discussed in Section IV. The reformulation of Ampere's law is

$$\nabla \times \vec{H} = 4\pi \vec{J} + \frac{\partial}{\partial t}\left(\vec{D}_C + \epsilon_r \vec{D}_F\right). \tag{63}$$

Ampere's law must be written in the above form to be able to derive the wave equation, as explained in Section IX. Other alternatives to the above equation will be considered in Section XII, but these will be shown to be incorrect.

Applying the $\nabla \cdot$ operation on both sides of the above equation,

$$\nabla \cdot \left(\nabla \times \vec{H}\right) = \nabla \cdot \left(4\pi \vec{J}\right) + \frac{\partial}{\partial t}\nabla \cdot \left(\vec{D}_C \ + \epsilon_r \vec{D}_F\right). \tag{64}$$

From calculus, the left-hand side of the above equation is 0, and to be able to derive the current-continuity equation, Gauss's law must satisfy,

$$\nabla \cdot \left(\vec{D}_C + \epsilon_r \vec{D}_F\right) = 4\pi\rho. \tag{65}$$

The reformulated Maxwell's equations are

$$\nabla \cdot \left(\vec{D}_C + \epsilon_r \vec{D}_F\right) = 4\pi\rho \tag{66}$$

$$\nabla \cdot \vec{B} = 0 \tag{67}$$

$$\nabla \times \vec{D}_F = -\frac{\partial \vec{B}}{\partial t} \tag{68}$$

$$\nabla \times \vec{E}_C = \vec{0} \tag{69}$$

$$\nabla \times \vec{H} = 4\pi \vec{J} + \frac{\partial}{\partial t}\left(\vec{D}_C + \epsilon_r \vec{D}_F\right), \tag{70}$$

and repeating Equation 56 – Equation 59,

$$\vec{D}_F = \epsilon_r \vec{E}_F \tag{71}$$

$$\vec{D}_C = \epsilon_r \vec{E}_C \tag{72}$$

$$\vec{B} = \mu_o \mu_r \vec{H} \tag{73}$$

$$\mu_o = \frac{1}{c^2}, c \approx 3.0 \times 10^{10}\ cm/s. \tag{74}$$

These equations are also valid for time-varying fields and time-varying sources. The electric displacement and electric field at any point are

$$\vec{D} = \vec{D}_C + \vec{D}_F \tag{75}$$

$$\vec{E} = \vec{E}_C + \vec{E}_F. \tag{76}$$

It can be noted in the reformulation, in modifying $\vec{E}_F$ to $\vec{D}_F = \epsilon_r \vec{E}_F$ in Faraday's law, ϵ_r has been "relocated" to the factor of $\vec{D}_F$ in Gauss's law and Ampere's law. Note the symmetry of Equations 66-70 to the equations in the current formulation in Equations 51-55.

IX. Derivation of the Wave Equation in the Reformulated Maxwell's Equations

In a source-free region, and a uniform medium with permittivity ϵ_r and permeability μ_r, sufficiently far away that $\vec{D}_C$ has decayed to 0, the revised Ampere's law and Faraday's law are

$$\nabla \times \vec{D}_F = -\frac{\partial \vec{B}}{\partial t} \tag{77}$$

$$\nabla \times \vec{H} = \epsilon_r \frac{\partial \vec{D}_F}{\partial t}. \tag{78}$$

In addition, if a uniform material is assumed, Equation 66 and Equation 67 reduce to

$$\nabla \cdot \vec{D}_F = 0 \tag{79}$$

$$\nabla \cdot \vec{H} = 0, \tag{80}$$

where $\vec{D}_C = 0$ has been assumed to derive Equation 79. Using the identity,

$$\nabla \times \nabla \times \vec{R} = \nabla\left(\nabla \cdot \vec{R}\right) - \nabla^2 \vec{R}, \tag{81}$$

where $\vec{R}$ is a vector field with the above operations defined,

$$\nabla \times \nabla \times \vec{D}_F = \nabla\left(\nabla \cdot \vec{D}_F\right) - \nabla^2 \vec{D}_F. \tag{82}$$

Simplifying the above equation using Equation 79,

$$\nabla \times \nabla \times \vec{D}_F = -\nabla^2 \vec{D}_F. \tag{83}$$

From Equation 73-74 and Equation 77,

$$\nabla \times \nabla \times \vec{D}_F = -\frac{\mu_r}{c^2}\frac{\partial}{\partial t}\left(\nabla \times \vec{H}\right). \tag{84}$$

Since a uniform dielectric medium is assumed, μ_r can be factored out of the $\nabla \times$ operation. Substituting Equation 78 in the above expression,

$$\nabla \times \nabla \times \vec{D}_F = -\frac{\mu_r \epsilon_r}{c^2}\frac{\partial^2 \vec{D}_F}{\partial t^2}. \tag{85}$$

From Equation 83 and Equation 85,

$$\nabla^2 \vec{D}_F = \frac{\mu_r \epsilon_r}{c^2}\frac{\partial^2 \vec{D}_F}{\partial t^2}, \tag{86}$$

is the wave equation. Using Equation 71, the above equation can be written as

$$\nabla^2 \vec{E}_F = \frac{\mu_r \epsilon_r}{c^2}\frac{\partial^2 \vec{E}_F}{\partial t^2}. \tag{87}$$

Similarly, the wave equation of the magnetic field $\vec{H}$ can also be derived.

X. Reformulated Equations in Electrostatics

In the case of electrostatics, where there are no time-varying sources and fields,

$$\vec{D}_F = \vec{0}, \tag{88}$$

since there is no time-varying $\vec{B}$ to generate a $\vec{D}_F$, as formulated in Faraday's law. Since there is no current flow, or magnetic materials to generate a magnetic field in electrostatics,

$$\vec{B} = \vec{0} \tag{89}$$

$$\vec{H} = \vec{0}. \tag{90}$$

Equation 66 – Equation 74 reduce to

$$\nabla \cdot \vec{D}_C = 4\pi\rho \tag{91}$$

$$\nabla \times \vec{E}_C = \vec{0} \tag{92}$$

$$\vec{D}_C = \epsilon_r \vec{E}_C, \tag{93}$$

where the fields and sources are not a function of time. Writing the revised set of equations using $\vec{E}_C$, $\vec{E}_F$, $\vec{D}_C$, $\vec{D}_F$, results in the proper formulation of the conservative property of the electric field in electrostatics, as written in Equation 92, unlike Equation 31.

XI. Reformulated Equations in Magnetostatics

In the case of magnetostatics, where there are no time-varying sources or time-varying fields, no electric charges, and only a steady current flow,

$$\vec{D}_C = \vec{0} \tag{94}$$

$$\vec{D}_F = \vec{0} \tag{95}$$

$$\vec{E}_C = \vec{0} \tag{96}$$

$$\vec{E}_F = \vec{0}. \tag{97}$$

Equation 66 – Equation 74 reduce to

$$\nabla \cdot \vec{B} = 0 \tag{98}$$

$$\nabla \times \vec{H} = 4\pi\vec{J} \tag{99}$$

$$\vec{B} = \mu_r \vec{H}, \tag{100}$$

and the fields and the sources do not vary over time.

XII. Alternative Formulations of Ampere's Law are Incorrect

Alternately, if Ampere's law is formulated as

$$\nabla \times \vec{H} = 4\pi\vec{J} + \frac{\partial}{\partial t}\left(\epsilon_r \vec{D}_C + \epsilon_r \vec{D}_F\right), \tag{101}$$

the wave equation can still be derived. However, this results in an inconsistency in the mathematical formulation. Applying the divergence $\nabla\cdot$ operation on both sides of the above equation,

$$\nabla \cdot \left(\nabla \times \vec{H}\right) = 4\pi\nabla \cdot \vec{J} + \frac{\partial}{\partial t}\nabla \cdot \left(\epsilon_r \vec{D}_C + \epsilon_r \vec{D}_F\right). \tag{102}$$

From calculus, the left-hand side of the above equation reduces to 0,

$$0 = 4\pi\nabla \cdot \vec{J} + \frac{\partial}{\partial t}\nabla \cdot \left(\epsilon_r \vec{D}_C + \epsilon_r \vec{D}_F\right). \tag{103}$$

This means that Gauss's law must be incorrrectly formulated as

$$\nabla \cdot \left(\epsilon_r \vec{D}_C + \epsilon_r \vec{D}_F\right) = 4\pi\rho, \tag{104}$$

to derive the current-continuity equation, as derived earlier. However, the above formulation of Gauss's law is incorrect. In the electrostatic case, $\vec{D}_F = 0$, and the above equation reduces to

$$\nabla \cdot \left(\epsilon_r \vec{D}_C\right) = 4\pi\rho. \tag{105}$$

However, the above equation contradicts the results of Faraday's experiment with spherical capacitors in electrostatics,

$$\nabla \cdot \vec{D}_C = 4\pi\rho. \tag{106}$$

Equation 101 is also incorrect for the reason mentioned in the remainder of this section. This shows other mathematical constraints that must be considered to obtain a proper formulation. In a source-free region, Equation 103 reduces to

$$0 = \frac{\partial}{\partial t}\nabla \cdot \left(\epsilon_r \vec{D}_C + \epsilon_r \vec{D}_F\right). \tag{107}$$

The above equation means that

$$\nabla \cdot \left(\epsilon_r \vec{D}_C + \epsilon_r \vec{D}_F\right) \tag{108}$$

must be a constant non time-varying value. It can be shown by proof by contradiction that the only possibility for the above term is 0,

$$\nabla \cdot \left(\epsilon_r \vec{D}_C + \epsilon_r \vec{D}_F\right) = 0. \tag{109}$$

Assume that the term in Equation 108 is a non-zero value. If the source of the field is moved far away, or waited sufficient time, so the field $\vec{D}_C$ and $\vec{D}_F$ decay to 0. In this case,

$$\nabla \cdot \left(\epsilon_r \vec{D}_C + \epsilon_r \vec{D}_F\right) = 0. \tag{110}$$

This contradicts the initial assumption that the term in Equation 108 is a non-zero value. Therefore, the only possibility is that

$$\nabla \cdot \left(\epsilon_r \vec{D}_C + \epsilon_r \vec{D}_F\right) = 0, \tag{111}$$

always holds true at all time. However, the above equation is inconsistent with Gauss's law in the source-free region,

$$\nabla \cdot \left(\vec{D}_C + \epsilon_r \vec{D}_F\right) = 0. \tag{112}$$

The above two equations cannot be satisifed at the same time. This shows that the only possibility of Ampere's law is Equation 70.

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