

Refutation of Tarski's geometric axioms and betweenness

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Abstract: Of 13 equations evaluated, five are tautologous and eight are *non* tautologous. This refutes Tarski's geometric axioms and betweenness which form a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬; + Or, ∨, ∪, ∪; - Not Or; & And, ∧, ∩, ∩, ; \ Not And;
 > Imply, greater than, →, ⇒, ↗, >, ⊃, ≻; < Not Imply, less than, ∈, <, ⊂, ≠, ≠, ≪, ≲;
 = Equivalent, ≡, :=, ⇔, ↔, ≅, ≈, ≅; @ Not Equivalent, ≠;
 % possibility, for one or some, ∃, ∃, M; # necessity, for every or all, ∀, □, L;
 (z=z) T as tautology, T, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥, zero;
 (%z>#z) N as non-contingency, Δ, ordinal 1; (%z<#z) C as contingency, ∇, ordinal 2;
 ~(y < x) (x ≤ y), (x ⊆ y); (A=B) (A~B).
 Note for clarity, we usually distribute quantifiers onto each designated variable.

From: en.wikipedia.org/wiki/Tarski%27s_axioms

[Alfred] Tarski's ... axiom set for the substantial fragment of Euclidean geometry ... is formulable in first-order logic with identity, and requiring no set theory ([from] 1959) (i.e., that part of Euclidean geometry that is formulable as an elementary theory).

Congruence axioms

Reflexivity of congruence $xy \equiv yx$ (1.1)

Remark 1.1: xy is not read point x is less than y as equivalent to point y is greater than x but rather $x \& y$ is equivalent to $y \& x$.

LET $p, q, r, s, t, u, v, w, x, y, z$: $a, b, d, e, u, v, w, x, y, z$.

$(x \& y) = (y \& x)$; TTTT TTTT TTTT TTTT (1.2)

Identity of congruence $xy \equiv zz \rightarrow x = y$ (2.1)

$((x \& y) = (z \& z)) > (x = y)$; TTTT TTTT TTTT TTTT (16)
FFFF FFFF FFFF FFFF (32)
 TTTT TTTT TTTT TTTT (80) (2.2)

Transivity of congruence $(xy \equiv zu \wedge xy \equiv vw) \rightarrow zu \equiv vw$ (3.1)

$((x \& y) = (z \& u)) \& ((x \& y) = (v \& w)) > ((z \& u) = (v \& w))$; TTTT TTTT TTTT TTTT (3.2)

Betweenness axioms

Identity of betweenness $B_{xyx} \rightarrow x=y$ (4.1)

$((y>x) \& (x>y)) > (x=y)$; TTTT TTTT TTTT TTTT (4.2)

Axiom of Pasch $(B_{xuz} \wedge B_{yvz}) \rightarrow \exists a (B_{uay} \wedge B_{vax})$ (5.1)

$((u>x) \& (z>u)) \& ((v>y) \& (z>v)) > (((\%p>u) \& (y>\%p)) \& ((\%p>v) \& (x>\%p)))$;
 NFNF NFNF NFNF NFNF (2) } x2
 TTTT TTTT TTTT TTTT (6) }
 FFFF FFFF FFFF FFFF (4) } x2
 TTTT TTTT TTTT TTTT (4) }
 FFFF FFFF FFFF FFFF (2) } x4
 TTTT TTTT TTTT TTTT (2) }
 FFFF FFFF FFFF FFFF (6) } x2
 CTCT CTCT CTCT CTCT (2) }
 TTTT TTTT TTTT TTTT (48)
 TTTT TTTT TTTT TTTT (6) } x2
 CTCT CTCT CTCT CTCT (2) } (5.2)

Axiom schema of continuity

$\exists a \forall x \forall y [(\varphi(x) \wedge \psi(y)) \rightarrow B_{axy}] \rightarrow \exists b \forall x \forall y [(\varphi(x) \wedge \psi(y)) \rightarrow B_{xby}]$ (11.1)

LET $u, v: \varphi, \psi$.

$((u \& \#x) \& (v \& \#y)) > ((\#x > \%p) \& (\#y > \#x)) > (((u \& \#x) \& (v \& \#y)) > ((\%q > \#x) \& (\#y > \%q)))$;
 TTTT TTTT TTTT TTTT (54)
 TCTT TCTT TCTT TCTT (2)
 TTTT TTTT TTTT TTTT (6)
 TCTT TCTT TCTT TCTT (2) (11.2)

Lower dimension $\exists a \exists b \exists c [\neg B_{abc} \wedge \neg B_{bca} \wedge \neg B_{cab}]$ (6.1)

$(\sim((\%q > \%p) \& (\%r > \%q)) \& \sim((\%r > \%q) \& (\%p > \%r))) \& \sim((\%p > \%r) \& (\%q > \%p))$;
 FFFF FFFF FFFF FFFF (6.2)

Congruence and betweenness

Upper dimension $(xu \equiv xy \wedge yu \equiv yv \wedge zu \equiv zv \wedge u \neq v) \rightarrow (B_{xyz} \vee B_{yzx} \vee B_{zxy})$ (7.1)

$((((x \& u) = (x \& v)) \& ((y \& u) = (y \& v))) \& (((z \& u) = (z \& v)) \& (u \neq v))) >$
 $((((y > x) \& (z > y)) \& ((z > y) \& (x > z))) \& ((x > z) \& (y > x)))$;
 TTTT TTTT TTTT TTTT (7.2)

Axiom of Euclid

A: $(B_{xyw} \wedge xy \equiv yw) \wedge (B_{xuv} \wedge xu \equiv uv) \wedge (B_{yuz} \wedge yu \equiv zu) \rightarrow yz \equiv vw$ (8.1.1)

$(((((y > x) \& (w > y)) \& (x \& y)) = (y \& w)) \& (((u > x) \& (v > u)) \& (x \& u)) = (u \& v))) \&$
 $((((u > y) \& (z > u)) \& (y \& u)) = (z \& u)) > ((y \& z) = (v \& w))$;
 TTTT TTTT TTTT TTTT (12)
 FFFF FFFF FFFF FFFF (2)
 TTTT TTTT TTTT TTTT (14)
 FFFF FFFF FFFF FFFF (4)
 TTTT TTTT TTTT TTTT (28)
 FFFF FFFF FFFF FFFF (2)

$$\begin{array}{l}
TTTT \ TTTT \ TTTT \ TTTT \ (14) \\
\mathbf{FFFF} \ \mathbf{FFFF} \ \mathbf{FFFF} \ \mathbf{FFFF} \ (2) \\
TTTT \ TTTT \ TTTT \ TTTT \ (14) \\
\mathbf{FFFF} \ \mathbf{FFFF} \ \mathbf{FFFF} \ \mathbf{FFFF} \ (2) \\
TTTT \ TTTT \ TTTT \ TTTT \ (2) \\
\mathbf{FFFF} \ \mathbf{FFFF} \ \mathbf{FFFF} \ \mathbf{FFFF} \ (6) \\
TTTT \ TTTT \ TTTT \ TTTT \ (18) \\
\mathbf{FFFF} \ \mathbf{FFFF} \ \mathbf{FFFF} \ \mathbf{FFFF} \ (2) \\
TTTT \ TTTT \ TTTT \ TTTT \ (6)
\end{array} \tag{8.1.2}$$

B: $Bxyz \vee Byzx \vee Bxzy \vee \exists a(xa \equiv ya \wedge xa \equiv za)$ (8.2.1)

$$\begin{array}{l}
(((y>x) \& (z>y)) \& ((z>y) \& (x>z))) \& ((x>z) \& (y>x))) \& (((x \& \%p) = (y \& \%p)) \& ((x \& \%p) = (z \& \%p))) ; \\
TTTT \ TTTT \ TTTT \ TTTT \ (16) \\
\mathbf{FFFF} \ \mathbf{FFFF} \ \mathbf{FFFF} \ \mathbf{FFFF} \ (96) \\
TTTT \ TTTT \ TTTT \ TTTT \ (16)
\end{array} \tag{8.2.2}$$

C: $(Bxuv \wedge Byuz \wedge x \neq u) \rightarrow \exists a \exists b (Bxya \neq Bxzb \wedge Bavb)$ (8.3.1)

$$\begin{array}{l}
(((u>x) \& (v>u)) \& ((u>y) \& (z>u))) \& (x@u) > \\
(((y>x) \& (\%p>y)) \& ((z>x) \& (\%q>z))) \& ((v>\%p) \& (\%q>v))) ; \\
TTTT \ TTTT \ TTTT \ TTTT \ (10) \\
TTTT \ TTTT \ TTTT \ TTTT \ (6) \ } \times 2 \\
\mathbf{NNFF} \ \mathbf{NNFF} \ \mathbf{NNFF} \ \mathbf{NNFF} \ (2) \ } \\
TTTT \ TTTT \ TTTT \ TTTT \ (6) \ } \\
\mathbf{NNFF} \ \mathbf{NNFF} \ \mathbf{NNFF} \ \mathbf{NNFF} \ (2) \ } \\
TTTT \ TTTT \ TTTT \ TTTT \ (16) \ } \\
TTTT \ TTTT \ TTTT \ TTTT \ (54)
\end{array} \tag{8.3.2}$$

Five segment $(x \neq y \wedge Bxyz \wedge Bx'y'z' \wedge xy \equiv x'y' \wedge yz \equiv y'z' \wedge xu \equiv x'u' \wedge yu \equiv y'u') \rightarrow zu \equiv z'u'$ (9.1)

LET $p, q, r, t, u, x, y, z:$
 $x', y', z', u', u, x, y, z$

$$\begin{array}{l}
(((x@y) \& (((y>x) \& (z>y)) \& ((q>p) \& (r>q)))) \& \\
(((x \& y) = (p \& q)) \& ((y \& z) = (q \& r))) \& (((x \& u) = (p \& t)) \& ((y \& u) = (q \& t)))) > ((z \& u) = (r \& t)) ; \\
TTTT \ TTTT \ TTTT \ TTTT
\end{array} \tag{9.2}$$

Segment construction $\exists z[Bxyz \wedge yz \equiv ab]$ (10.1)

LET $p, q:$ a, b

$$\begin{array}{l}
((y>x) \& (\#z>y)) \& ((y \& \#z) = (p \& q)) ; \\
\mathbf{TTTF} \ \mathbf{TTTF} \ \mathbf{TTTF} \ \mathbf{TTTF} \ (32) \\
\mathbf{FFFF} \ \mathbf{FFFF} \ \mathbf{FFFF} \ \mathbf{FFFF} \ (16) \\
\mathbf{TTTF} \ \mathbf{TTTF} \ \mathbf{TTTF} \ \mathbf{TTTF} \ (16) \\
\mathbf{CCCF} \ \mathbf{CCCF} \ \mathbf{CCCF} \ \mathbf{CCCF} \ (32) \\
\mathbf{FFFF} \ \mathbf{FFFF} \ \mathbf{FFFF} \ \mathbf{FFFF} \ (16) \\
\mathbf{CCCN} \ \mathbf{CCCN} \ \mathbf{CCCN} \ \mathbf{CCCN} \ (16)
\end{array} \tag{10.2}$$

Of 13 eqs. evaluated, five are tautologous and eight are *non* tautologous. This refutes Tarski's geometric axioms and betweenness.