Solution of a Vector-Triangle Problem Via Geometric (Clifford) Algebra

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Abstract

As a high-school-level application of Geometric Algebra (GA), we show how to solve a simple vector-triangle problem. Our method highlights the use of outer products and inverses of bivectors.

1 Introduction

Solving simple vector-triangle problems efficiently is an important skill to be developed at the pre-university level. The Geometric-Algebra (GA) concepts that we use here are discussed in greater detail in Refs. [1] and [2].

2 Problem Statement

"Given $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, *and* $\mathbf{c}$ *in Fig. 1, find* $\mathbf{a}$ *and* $\mathbf{b}$*."*

3 Solution

From Fig. 1,

$$\mathbf{a} + \mathbf{b} = \mathbf{c}. \tag{3.1}$$

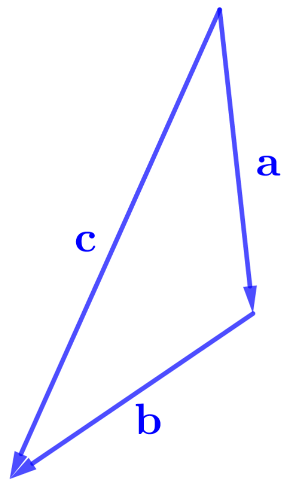

Figure 1: The vector triangle that we will solve.

We'll start with $\mathbf{a}$. Because we know $\hat{\mathbf{a}}$, we can find $\mathbf{a}$ as $\mathbf{a} = \|\mathbf{a}\|\hat{\mathbf{a}}$. Thus, in order to find $\|\mathbf{a}\|$, we'll rewrite Eq. (3.1) as

$$\|\mathbf{a}\|\hat{\mathbf{a}} + \mathbf{b} = \mathbf{c}.$$

Next, we'll eliminate $\mathbf{b}$ by taking the outer product of both sides with $\hat{\mathbf{b}}$:

$$\begin{aligned}
(\|\mathbf{a}\|\hat{\mathbf{a}} + \mathbf{b}) \wedge \hat{\mathbf{b}} &= \mathbf{c} \wedge \hat{\mathbf{b}} \\
\|\mathbf{a}\|\hat{\mathbf{a}} \wedge \hat{\mathbf{b}} + \underbrace{\mathbf{b} \wedge \hat{\mathbf{b}}}_{=0} &= \mathbf{c} \wedge \hat{\mathbf{b}} \\
\|\mathbf{a}\|\hat{\mathbf{a}} \wedge \hat{\mathbf{b}} &= \mathbf{c} \wedge \hat{\mathbf{b}}.
\end{aligned}$$

Finally, we multiply both sides by the inverse of $\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}$:

$$\begin{aligned}
\|\mathbf{a}\| \left(\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}\right) \left(\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}\right)^{-1} &= \left(\mathbf{c} \wedge \hat{\mathbf{b}}\right) \left(\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}\right)^{-1} \\
\|\mathbf{a}\| &= \left(\mathbf{c} \wedge \hat{\mathbf{b}}\right) \left(\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}\right)^{-1}. \qquad (3.2)
\end{aligned}$$

To find $\mathbf{b}$, we proceed similarly, by finding $\|\mathbf{b}\|$:

$$\begin{aligned}
\mathbf{a} + \mathbf{b} &= \mathbf{c} \\
\mathbf{a} + \|\mathbf{b}\|\hat{\mathbf{b}} &= \mathbf{c} \\
\left(\mathbf{a} + \|\mathbf{b}\|\hat{\mathbf{b}}\right) \wedge \hat{\mathbf{a}} &= \mathbf{c} \wedge \hat{\mathbf{a}} \\
\underbrace{\mathbf{a} \wedge \hat{\mathbf{a}}}_{=0} + \|\mathbf{b}\|\hat{\mathbf{b}} \wedge \hat{\mathbf{a}} &= \mathbf{c} \wedge \hat{\mathbf{a}} \\
\|\mathbf{b}\|\hat{\mathbf{b}} \wedge \hat{\mathbf{a}} &= \mathbf{c} \wedge \hat{\mathbf{a}} \\
\|\mathbf{b}\| \left(\hat{\mathbf{b}} \wedge \hat{\mathbf{a}}\right) \left(\hat{\mathbf{b}} \wedge \hat{\mathbf{a}}\right)^{-1} &= (\mathbf{c} \wedge \hat{\mathbf{a}}) \left(\hat{\mathbf{b}} \wedge \hat{\mathbf{a}}\right)^{-1} \\
\|\mathbf{b}\| &= (\mathbf{c} \wedge \hat{\mathbf{a}}) \left(\hat{\mathbf{b}} \wedge \hat{\mathbf{a}}\right)^{-1}.
\end{aligned}$$

4 Comments

References

[1] A. Macdonald, *Linear and Geometric Algebra* (First Edition) p. 126, CreateSpace Independent Publishing Platform (Lexington, 2012).

[2] J. Smith, 2016, "Some Solution Strategies for Equations that Arise in Geometric (Clifford) Algebra", http://vixra.org/abs/1610.0054.