Solution of a Vector-Triangle Problem Via Geometric (Clifford) Algebra

May 15, 2019

James Smith
nitac14b@yahoo.com
https://mx.linkedin.com/in/james-smith-1b195047

Abstract

As a high-school-level application of Geometric Algebra (GA), we show how to solve a simple vector-triangle problem. Our method highlights the use of outer products and inverses of bivectors.

1 Introduction

Solving simple vector-triangle problems efficiently is an important skill to be developed at the pre-university level. The Geometric-Algebra (GA) concepts that we use here are discussed in greater detail in Refs. [1] and [2].

2 Problem Statement

“Given \( \hat{a}, \hat{b}, \) and \( c \) in Fig. 1, find \( a \) and \( b \).”

3 Solution

From Fig. 1

\[ a + b = c. \] (3.1)
We’ll start with $\mathbf{a}$. Because we know $\hat{\mathbf{a}}$, we can find $\mathbf{a}$ as $\mathbf{a} = \|\mathbf{a}\| \hat{\mathbf{a}}$. Thus, in order to find $\|\mathbf{a}\|$, we’ll rewrite Eq. (3.1) as

$$\|\mathbf{a}\| \hat{\mathbf{a}} + \mathbf{b} = \mathbf{c}.$$  

Next, we’ll eliminate $\mathbf{b}$ by taking the outer product of both sides with $\hat{\mathbf{b}}$:

$$(\|\mathbf{a}\| \hat{\mathbf{a}} + \mathbf{b}) \wedge \hat{\mathbf{b}} = \mathbf{c} \wedge \hat{\mathbf{b}}$$

$$\|\mathbf{a}\| \hat{\mathbf{a}} \wedge \hat{\mathbf{b}} + \mathbf{b} \wedge \hat{\mathbf{b}} = \mathbf{c} \wedge \hat{\mathbf{b}}$$

$$\|\mathbf{a}\| \hat{\mathbf{a}} \wedge \hat{\mathbf{b}} = \mathbf{c} \wedge \hat{\mathbf{b}}.$$  

Finally, we multiply both sides by the inverse of $\hat{\mathbf{a}} \wedge \hat{\mathbf{b}}$:

$$\|\mathbf{a}\| \left( \hat{\mathbf{a}} \wedge \hat{\mathbf{b}} \right) \left( \hat{\mathbf{a}} \wedge \hat{\mathbf{b}} \right)^{-1} = \left( \mathbf{c} \wedge \hat{\mathbf{b}} \right) \left( \hat{\mathbf{a}} \wedge \hat{\mathbf{b}} \right)^{-1}$$

$$\|\mathbf{a}\| = \left( \mathbf{c} \wedge \hat{\mathbf{b}} \right) \left( \hat{\mathbf{a}} \wedge \hat{\mathbf{b}} \right)^{-1}. \quad (3.2)$$

To find $\mathbf{b}$, we proceed similarly, by finding $\|\mathbf{b}\|$: 

$$\mathbf{a} + \mathbf{b} = \mathbf{c}$$

$$\mathbf{a} + \|\mathbf{b}\| \hat{\mathbf{b}} = \mathbf{c}$$

$$\left( \mathbf{a} + \|\mathbf{b}\| \hat{\mathbf{b}} \right) \wedge \hat{\mathbf{a}} = \mathbf{c} \wedge \hat{\mathbf{a}}$$

$$\mathbf{a} \wedge \hat{\mathbf{a}} + \|\mathbf{b}\| \hat{\mathbf{b}} \wedge \hat{\mathbf{a}} = \mathbf{c} \wedge \hat{\mathbf{a}}$$

$$\|\mathbf{b}\| \hat{\mathbf{b}} \wedge \hat{\mathbf{a}} = \mathbf{c} \wedge \hat{\mathbf{a}}$$

$$\|\mathbf{b}\| \left( \hat{\mathbf{b}} \wedge \hat{\mathbf{a}} \right) \left( \hat{\mathbf{b}} \wedge \hat{\mathbf{a}} \right)^{-1} = \left( \mathbf{c} \wedge \hat{\mathbf{a}} \right) \left( \hat{\mathbf{b}} \wedge \hat{\mathbf{a}} \right)^{-1}$$

$$\|\mathbf{b}\| = \left( \mathbf{c} \wedge \hat{\mathbf{a}} \right) \left( \hat{\mathbf{b}} \wedge \hat{\mathbf{a}} \right)^{-1}.$$
4 Comments

References
