Refutation of intutionistic Zermelo-Fraenkel set theory (IZF), de Jongh's theorem, and CZF

Abstract: We evaluate the nine axioms for intutionistic Zermelo-Fraenkel set theory (IZF). None is tautologous. This refutes IZF and its use in blended models and denies De Jongh's classical theorem and similar results for constructive ZF (CZF). These segments form a *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, **F** as contradiction, N as truthity (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·; \ Not And;
> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ↠ ; < Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇐⇒, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) T as tautology, ⊤, ordinal 3; (z@z) **F** as contradiction, Ø, Null, ⊥, zero;
(%z>#z) N as non-contingency, Δ, ordinal 1; (%z<#z) C as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y); (A=B) (A~B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: Passmann, R. (2019). De Jongh's theorem for intutionistic Zermelo-Fraenkel set theory. arxiv.org/pdf/1905.04972.pdf robertpassmann@posteo.de

Abstract. We prove that the propositional logic of intuitionistic set theory IZF is intuitionistic propositional logic IPC. More generally, we show that IZF has the de Jongh property with respect to every intermediate logic that is complete with respect to a class of finite trees. The same results follow for CZF.

1. Introduction: De Jongh's classical theorem ... states that the propositional logic of Heyting Arithmetic HA is intuitionistic logic IPC. In this work, we will prove de Jongh's theorem for intuitionistic Zermelo-Fraenkel set theory IZF. … To prove this result, we introduce a new semantics for IZF, the so-called *blended Kripke models*, or *blended models* for short.

3. Blended models: In this section, we will construct the blended models and show that they are models of intuitionistic Zermelo-Fraenkel set theory IZF. Figure 1. The axioms of IZF.

Extensionality: ∀a∀b(∀x(x ∈ a ↔ x ∈ b) → a = b) (3.1.1)

LET a, b, x: p, q, r

((#r<#p)=(#p<#q))>(#p=#q) ; TTCT TTCT TTCT TTCT (3.2.1)

Empty set: ∃a ∀x ∈ a ⊥ (3.2.1)

(%p&#r)<(p&(p@p)) ; **FFFF** **F**N**F**N **FFFF** **F**N**F**N (3.2.2)

Pairing: $\forall a \forall b \exists y \forall x(x \in y \leftrightarrow (x = a \lor x = b))$ (3.3.1)

(#r<%s)=((#r=#p)+(#r=#q)) ; **FFF**N **F**NNN **FFF**N **F**NNN (3.3.2)

Remark 3.3.2: If the quantifiers are not distributed, Eq. 3.3.1 becomes a contradiction.

((#p&#q)&(%s&#r))&((r<s)=((r=p)+(r=q))) ;
FFFF FFFF FFFF FFFF (3.3.3)

Union: $\forall a \exists y \forall x(x \in y \leftrightarrow \exists u(u \in a \land x \in u))$ (3.4.1)

LET p, q, r, s: a, u, x, y

(#r<%s)=((%q<#p)&(#r<%q)) ; TTTT CCCC TTTT CCCC (3.4.2)

Power set: $\forall a \exists y \forall x(x \in y \leftrightarrow x \subseteq a)$(3.5.1)

(#r<%s)=~(#p<#r); **F**N**F**N NNNN **F**N**F**N **FFFF** (3.5.2)

Infinity: $\exists a(\exists x\ x \in a \land \forall x \in a \exists y \in a\ x \in y)$ (3.6.1)

((%r<%p)&(#r<%p))&((%s<%p)&(#r<%s));
FFFF FFFF FFFF FFFF (3.6.2)

Remark 3.6.2: Eq. 3.6.2 as rendered is *not* tautologous and a contradiction.

Set Induction: $(\forall a(\forall x \in a\ \phi(x) \rightarrow \phi(a))) \rightarrow \forall a \phi(a)$, for all formulas $\phi(x)$. (3.7.1)

LET p, q, r, s: a, ϕ, x, y

(((#r<#p)&(#q&#r))>(q&#p))>(q&#p) ;
FFFN **FFF**N **FFF**N **FFF**N (3.7.2)

Separation: $\forall a \exists y \forall x(x \in y \leftrightarrow (x \in a \land \phi(x)))$, for all formulas $\phi(x)$. (3.8.1)

(#r<%s)=((#r<#p)&(#q&#r)) ; TTTT CCTC TTTT TTCT (3.8.2)

Collection: $\forall a(\forall x \in a \exists y\ \phi(x, y) \rightarrow \exists b \forall x \in a \exists y \in b\ \phi(x, y))$, for all formulas $\phi(x, y)$, where b is not free in $\phi(x, y)$. (3.9.1)

LET p, q, x, y, ϕ: a, b, x, y, z

((#x<#p)&(%y&(#z&(x&%y))))>(((%q&#x)<(#p&%y))<(q&(#z&(x&%y)))) ;
TTTT TTTT TTTT TTTT(112),
TCTC TCTC TCTC TCTC(16) (3.9.2)

Eqs. 3.1.2-3.9.2 are *not* tautologous. This refutes IZF and its use in blended models, De Jongh's classical theorem, and similar results for constructive ZF (CZF).