Noether's symmetry in Bosonic String σ-Model of (p=1)-brane Polyakov action under Poincare transformation

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Abstract

It is shown using very few steps that under Poincare transformation, the Polyakov action is invariant and thus, globally symmetric. It is only over this argument that the local symmetries of bosonic strings such as diffeomorphism and Weyl symmetry can be discussed.

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1. Introduction

The relativistic differential action is given by constant times the path length. It is expressed in terms of proper time using the metric elaboration and further in terms of an auxiliary field $e(\tau)$ in 1 and 2 respectively. A generalized p-brane action is expressed in 3, where $G_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}$.

$$S = -m \int d\tau \sqrt{-g_{\mu\nu} dX^\mu dX^\nu} \tag{1}$$

$$S = \frac{1}{2} \int d\tau [e(\tau)^{-1} \dot{X}^2 - m^2 e(\tau)] \tag{2}$$

$$S = -T \int d\sigma \sqrt{-det(G_{\alpha\beta})} d^{p+1} \tag{3}$$

The confinement of 3 to $(p = 1)$ brane configuration to obtain Nambu-Goto action can be performed by considering $\sigma^0 = \tau$ as time-like and $\sigma^1 = \sigma$

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as space-like. Correspondingly in Minkowski background, $G_{00} = \dot{X}^2$, $G_{01} = G_{10} = \dot{X}X'$ and $G_{11} = X'^2$.

$$S = -\frac{T}{2}\int d\tau d\sigma\sqrt{(\dot{X}X')^2 - (\dot{X}^2)(X'^2)} \tag{4}$$

With the same motivation to introduce an auxiliary field in 2 that the action turns out to be non-zero for mass-less strings, the Polyakov action is formulated in 5, where $h = det(h_{\alpha\beta})$.

$$S = -\frac{T}{2}\int d\tau d\sigma\sqrt{-h}h^{\alpha\beta}\frac{\partial X^\mu}{\partial\alpha}\frac{\partial X^\nu}{\partial\beta}g_{\mu\nu} \tag{5}$$

Noether's condition $\delta S = 0$ is shown to hold true by evaluating the action under the Poincare transformed basis by the rule (1) $\delta X^\mu(\tau,\sigma) = a^\mu_\nu X^\mu(\tau,\sigma) + b^\mu$ and (2) $\delta h^{\alpha\beta} = 0$, where a^μ_ν and b^μ can be inferred as the elements responsible for rotation and translation respectively.

2. The nature of rotation element a^μ_ν

Let us consider the operations $\tau' = cos(\psi)\tau + sin(\psi)\sigma$ and $\sigma' = -sin(\psi)\tau + cos(\psi)\sigma$. Using small angle approximations that $cos(\psi) \to 1$ and $sin(\psi) \approx \psi$, the operations narrow down to $\tau' = \tau + \psi\sigma$ and $\sigma' = -\psi\tau + \sigma$. We see that $a^1_2 = \psi$, $a^2_1 = \psi$ and rest a^μ_ν are zero, which makes a^μ_ν anti-symmetric. Since $a_{\mu\nu}$ will be used in section 4, it is also realized to be anti-symmetric by engaging Minkowski background to deduce $a_{12} = \eta_{1\lambda}a^\lambda_2 = \psi$ and $a_{21} = \eta_{2\lambda}a^\lambda_1 = -\psi$.

$$\therefore a_{\mu\nu} = -a_{\nu\mu} \tag{6}$$

3. The nature of auxiliary field $h^{\alpha\beta}$

Let us start by finding the determinant of $h^{\alpha\beta}$. It can be done by navigating through each element and multiplying the rest of the matrix by it, and then dividing by n!. Correspondingly, δh is found in 8 using 7.

$$det(h_{\alpha\beta}) = \frac{1}{n!}\epsilon^{\alpha_1...\alpha_n}\epsilon^{\beta_1...\beta_n}h_{\alpha_1\beta_n}...h_{\alpha_n\beta_n} \tag{7}$$

$$det(h_{\alpha\beta}) = \frac{1}{n!}\epsilon^{\alpha_1...\alpha_n}\epsilon^{\beta_1...\beta_n}h_{\alpha_1\beta_n}...h_{\alpha_n\beta_n} + \frac{1}{(n-1)!}\epsilon^{\alpha_1...\alpha_n}\epsilon^{\beta_1...\beta_n}h_{\alpha_2\beta_n}...h_{\alpha_n\beta_n} \tag{8}$$

We see from 8 that $\delta h = h^{\alpha\beta}\delta h_{\alpha\beta} h = -h^{\alpha\beta}\delta h_{\alpha\beta} h$. Therefore, the auxiliary field is symmetric.

$$h^{\alpha\beta} = h^{\beta\alpha} \tag{9}$$

4. Variation in Polyakov action for the transformed X^{μ}

$\delta X^{\mu}(\tau,\sigma) = a^{\mu}_{X}{}^{\mu}(\tau,\sigma) + b^{\mu}$, which is the Poincare transformation, is substituted in 5 to start with. This transformation will not alter $h^{\alpha\beta}$ as $\delta h = 0$ (defined in introduction).

$$\delta S = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}\, h^{\alpha\beta}\partial_{\alpha}(a^{\mu}_{\lambda}X^{\lambda} + b^{\mu})\partial_{\beta}X^{\nu} g_{\mu\nu} \tag{10}$$

Since a^{μ}_{λ} and b^{μ} are space-time independent, they can be extracted out from the brackets to yield 11. Irrespective of the nature of these terms, we realize that translation doesn't contribute in varying action. Thus, Poincare transformation reduces to Lorentz transformation. Further, a^{μ}_{λ} can be compressed with the metric to yield 12.

$$\delta S = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}\, h^{\alpha\beta} a^{\mu}_{\lambda}\partial_{\alpha}X^{\lambda}\partial_{\beta}X^{\nu} g_{\mu\nu} \tag{11}$$

$$\delta S = -\frac{T}{2}\int d\tau d\sigma \sqrt{-h}\, h^{\alpha\beta} a_{\nu\lambda}\partial_{\alpha}X^{\lambda}\partial_{\beta}X^{\nu} \tag{12}$$

5. Deduction

We see from 12 that since $a_{\nu\lambda}$ is anti-symmetric (shown in 6) and $h^{\alpha\beta}$ is symmetric (shown in 9), and that a symmetric term times an anti-symmetric term is 0, Noether's symmetry condition $\delta S = 0$ comes up.[2]

[2]Here, I would like to admire Emmy Noether for her work over which many theories in modern physics including the Standard Model stand. The theorem can be thought more as a validity providing test for the new theories - which in this case - has provided the same for bosonic string theory.