The Burnside $\mathbb{Q}$-algebras of a monoid

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To each monoid $M$ we attach an inclusion $A \hookrightarrow B$ of $\mathbb{Q}$-algebras, and ask: Is $B$ flat over $A$? If our monoid $M$ is a group, $A$ is von Neumann regular, and the answer is trivially Yes in this case.

In this text "$\mathbb{Q}$-algebra" means "associative commutative $\mathbb{Q}$-algebra with one".

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Let us define $A$.

Say that an $M$-set $X$ is indecomposable if $X \neq \emptyset$ and if $X$ is not a disjoint union of two nonempty sub-$M$-sets.

Let $\Xi$ be a set of finite indecomposable $M$-sets such that any finite indecomposable $M$-set is isomorphic to a unique $X \in \Xi$.

If $X, Y$ are in $\Xi$, then their product $X \times Y$ is a disjoint union $Z_1 \sqcup \cdots \sqcup Z_n$ of finite indecomposable $M$-sets. Moreover, if $Z \in \Xi$, then the number of $i$ such that $Z_i \cong Z$ is a nonnegative integer $m(X, Y, Z)$ which depends only on the isomorphism classes of $X, Y$ and $Z$.

We define $A$ as the $\mathbb{Q}$-vector space with basis $\Xi$ and multiplication given by

$$XY := \sum_{Z \in \Xi} m(X, Y, Z) \ Z.$$

In particular $A$ is a $\mathbb{Q}$-algebra.

We temporarily denote $A$ by $A(M)$ and $\Xi$ by $\Xi(M)$ to emphasize the dependence on $M$.

**Theorem 1.** The $\mathbb{Q}$-algebra $A(G)$ of a group $G$ is von Neumann regular.
Proof. If \( b \) is in \( A(G) \), then there is a largest finite index normal subgroup \( N \) of \( G \) such that \( b \in A(G/N) \). Let \( \phi_{G/N} : A(G/N) \to \mathbb{Q}^\Xi(G/N) \) be the \( \mathbb{Q} \)-algebra isomorphism defined in Section 3.3 of [1], and define \( b' \in A(G/N) \subset A(G) \) by
\[
b' = (\phi_{G/N})^{-1}\left(w \circ (\phi_{G/N}(b))\right),
\]
where \( w : \mathbb{Q} \to \mathbb{Q} \) is defined by \( w(\lambda) = \frac{1}{\lambda} \) if \( \lambda \neq 0 \) and \( w(0) = 0 \) (that is, \( w \) is a witness to the von Neumann regularity of \( \mathbb{Q} \)), so that we have \( b^2b' = b \) in \( A(G) \), which shows that \( A(G) \) is von Neumann regular. (Here \( X \subset Y \) means "\( X \) is a (not necessarily proper) subset of \( Y \).") \( \square \)

We denote again by \( \Xi \) and \( A \) (instead of \( \Xi(M) \) and \( A(M) \)) the set and the \( \mathbb{Q} \)-algebra defined above.

Let us define \( B \).

**Proposition 2.** For any \( Z \in \Xi \) there are only finitely \( (X, Y) \in \Xi^2 \) such that \( m(X, Y, Z) \) is nonzero.

Proof. It suffices to show that, for \( X, Y \in \Xi \) and \( Z \) an indecomposable component of \( X \times Y \), the projection \( p : X \times Y \to X \) maps \( Z \) onto \( X \). (Indeed, up to isomorphism, there are only finitely many quotients of \( Z \).)

Let us fix an element \( a \) of \( M \). Say that a point of an \( M \)-set is periodic if it is a fixed point of \( a^n \) for some \( n \geq 1 \).

The following facts are clear:

(a) If \( v \) is a periodic point of an \( M \)-set \( U \) and \( n \) is a nonnegative integer, then \( v = a^n u \) for some \( u \in U \).

(b) If \( u \) is a point of a finite \( M \)-set, then \( a^n u \) is periodic for \( n \) large enough.

Let \( p : X \times Y \to X \) be the projection, and assume by contradiction that \( p(Z) \) is a proper subset of \( X \). Then there is a tuple \( (a, x_1, x_2, y_2) \) with
\[
a \in M; \ x_1, x_2 \in X; \ x_1 \not\in p(Z); \ ax_1 = x_2; \ y_2 \in Y; \ (x_2, y_2) \in Z.
\]
It suffices to show \( x_1 \in p(Z) \). By (b) we can pick an \( n \in \mathbb{N} \) such that \( a^n(x_2, y_2) \in Z \) is periodic. Set
\[
x_3 := a^n x_2 = a^{n+1} x_1, \ y_3 := a^n y_2.
\]
By (a) there is a \( y_1 \in Y \) such that \( a^{n+1} y_1 = y_3 \), and we get
\[
a^{n+1}(x_1, y_1) = (x_3, y_3) \in Z,
\]
which implies \((x_1, y_1) \in Z\) and thus \(x_1 \in p(Z)\), contradiction. This completes the proof. □

Proposition 2 implies that the multiplication we defined above on \(A\) extends to the \(\mathbb{Q}\)-vector space of all expressions of the form

\[
\sum_{X \in \Xi} a_X X
\]

with \(a_X \in \mathbb{Q}\). We denote by \(B\) the \(\mathbb{Q}\)-algebra obtained by this process.

**Question 3.** Is \(B\) flat over \(A\)?

Beside the case of groups, there is only one case where I know that the answer is Yes. It is the case of the monoid \(M := \{0, 1\}\) with the obvious multiplication. In the post

[https://math.stackexchange.com/a/3154240/660](https://math.stackexchange.com/a/3154240/660)

Eric Wofsey shows the isomorphism \(A \cong \mathbb{Q}[x_1, x_2, \ldots]\), where the \(x_i\) are indeterminates, and it is clear that we have \(B \cong \mathbb{Q}[[x_1, x_2, \ldots]]\).

**Proposition 4.** The ring \(\mathbb{Q}[[x_1, x_2, \ldots]]\) is flat over \(\mathbb{Q}[x_1, x_2, \ldots]\).

The proof of Proposition 4 will use two lemmas:

**Lemma 5.** If \(A\) is a commutative ring with one, if \((M_i)_{i \in I}\) is a filtered inductive system of \(A\)-modules, and if \(N \to P\) is a morphism of \(A\)-modules, then the natural morphisms

\[
\text{colim} \ker(M_i \otimes_A N \to M_i \otimes_A P)
\]

\[
\to \ker \left( \text{colim}(M_i \otimes_A N) \to \text{colim}(M_i \otimes_A N) \right)
\]

\[
\to \ker \left( (\text{colim} M_i) \otimes_A N \to (\text{colim} M_i) \otimes_A N \right)
\]

are bijective.

**Proof.** This follows respectively from Lemmas 4.19.2

[https://stacks.math.columbia.edu/tag/002W](https://stacks.math.columbia.edu/tag/002W)

and 10.11.9

[https://stacks.math.columbia.edu/tag/00DD](https://stacks.math.columbia.edu/tag/00DD)

of [2]. □

**Lemma 6.** Filtered colimits preserve flatness. More precisely, if \(A\) and \((M_i)_{i \in I}\) are as above, and if in addition \(M_i\) is flat for all \(i\), then \(\text{colim} M_i\) is flat.
Proof. This follows immediately from Lemma 5. □

Proof of Proposition 4. We claim:
(a) $\mathbb{Q}[[x_1, x_2, \ldots]]$ is flat over $\mathbb{Q}[x_1, \ldots, x_n]$.
(b) Claim (a) implies the proposition.
Proof of (b). Set
$$A_n := \mathbb{Q}[[x_1, x_2, \ldots]] \otimes_{\mathbb{Q}[x_1, \ldots, x_n]} \mathbb{Q}[x_1, x_2, \ldots].$$
The ring $A_n$ being flat over $\mathbb{Q}[x_1, x_2, \ldots]$ and $\mathbb{Q}[[x_1, x_2, \ldots]]$ being the colimit of the $A_n$, Claim (b) follows from Lemma 6.

Proof of (a). The ring $\mathbb{Q}[[x_1, \ldots, x_n]]$ being noetherian by Lemma 10.30.2
https://stacks.math.columbia.edu/tag/036
of [2], and flat over $\mathbb{Q}[x_1, \ldots, x_n]$ by Lemma 10.96.2(1)
https://stacks.math.columbia.edu/tag/00MB
of [2], it is enough to verify that $\mathbb{Q}[[x_1, x_2, \ldots]]$ is flat over $\mathbb{Q}[[x_1, \ldots, x_n]]$.
But, since $\mathbb{Q}[[x_1, x_2, \ldots]]$, viewed as an $\mathbb{Q}[[x_1, \ldots, x_n]]$-module, is just a product of copies of $\mathbb{Q}[[x_1, \ldots, x_n]]$, it is flat over $\mathbb{Q}[[x_1, \ldots, x_n]]$ by Lemma 10.89.5
https://stacks.math.columbia.edu/tag/05CY
and Proposition 10.89.6
https://stacks.math.columbia.edu/tag/05CZ
of [2], we are done. □

References.
[1] Serge Bouc, Burnside rings, Chapter 1, pages 739-804, in Handbook of Algebra, Volume 2, 2000, doi 0.1037/a0028240
https://tinyurl.com/y6trypqv

Tex file available at
https://tinyurl.com/y5skaqim and https://tinyurl.com/y5jfbv5r

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