Classical mechanics reloaded

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Abstract : *The first postulate of the classical mechanics, stating that the position and the time are independent, is demonstrated as false, and replaced by a theorem stating that the position and the time are always related by a bijection, accordingly to the experiment. Introducing this theorem instead of the first postulate inside the calculus of variation, provides new equations of motion, close to those of Lagrange, but giving more information on the allowed trajectories. The velocity of a classical mobile appears as the addition of one or many of only two elementary uniform velocities, of rotation and translation, in a typical Fourier series fashion. The addition of a single elementary rotation and a single elementary translation, leads to the Keplerian motion, as expected. This approach can be used for any physical parameter, an illustration is given by the forecast of the Boltzmann's entropy, the ideal gas law and the equations driving the chemical kinetics.*

Introduction

The theoretical study of the classical mechanics has nearly vanished during the last century. The main opinion is that it has been fully investigated in the XVIIIth and XIXth centuries and nothing new could come from there.

We observe however that any plot with respect to the time of a classical physical parameter, can be split into a succession of intervals, in which the value of the parameter and the time are always related by a bijection (see figure 1). If this parameter is the position, it means that the position and the time are related by a bijection, at any time and any place on the trajectory. And this is a big problem because the first postulate of the classical mechanics pretends that the position and the time should be independent.

We then decided to investigate what happens when we replace the first postulate by a theorem stating the true structure of the trajectories, inside the classical mechanics machinery, i.e. mainly inside the variation of the action, leading to the equations of motion.

We will show that the variation of the action does not need the postulate of least action any more to get the equivalent of the Lagrange's equations of motion. These equivalents provide however an advantage : they give more information about the trajectories that are geometrically allowed.

In the case of classical systems, which momentum is their mass multiplied by their velocity, we will show that only two elementary motions are allowed : the uniform rotation, and the uniform translation. Any classical motion is then the superposition of one or many of these two elementary uniform motions, in a typical Fourier series fashion. Actually we demonstrate that any classical velocity must be described by its Fourier series, which terms are each representing a sub-velocity.

One of the most simple addition of motions that we can think of is the superposition of a single uniform rotation velocity, with a single uniform translation velocity. A system having such a velocity will exhibit a Keplerian trajectory, as expected from theorem of the Keplerian kinematics [2].

This new approach of the classical mechanics does not concern only the position in three dimensions, but actually it can apply to any physical parameter measured with respect to the time. In one dimension we even get the mathematical structure of the Lagrangian, so we do not have to postulate it. It appears that this Lagrangian is consistent with the formulation of the energy in thermodynamics, and also with the chemical kinetics.

It is important to note that in this work we will use no postulate, nor hypothesis, nor unprovable theory, but we will at contrary remove two postulates from the foundation of the classical mechanics. We will first define an obvious theorem, and second look at its consequences on the calculus of variation described in the literature. Nothing else. We do not pretend to any new theory of physics, but just to enforce the basement of the classical mechanics.

Non validity of the first postulate of classical mechanics

L. Landau and E. Lifchitz in their remarkable course "Mechanics"[1] explain that the Lagrangian L is the function that describes all the physical properties of a system. From scratch we do not know the mathematical structure of L, but we have to consider that it must depend upon three independent variables : the time t, the position $\mathbf{r}$ and the velocity $\mathbf{v}$. We can write $L=L(t,\mathbf{r},\mathbf{v})$. This is the first postulate of the classical mechanics.

However, looking at the experimental measures of the real trajectories, we must admit that they can always be split into successive intervals in which the position and the time are related by a bijection, as illustrated on the figure 1.

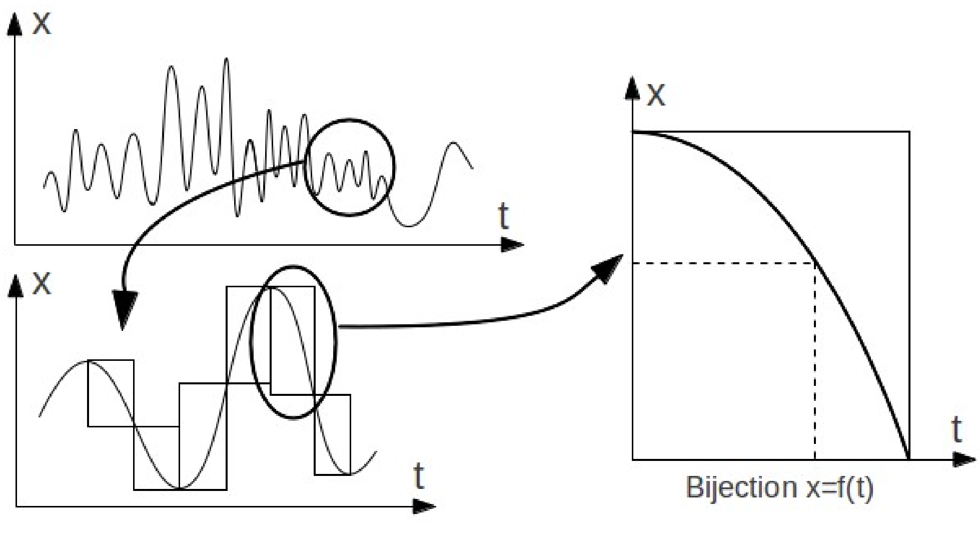

Figure 1 : *The trajectory of a classical mobile, recording its position x with respect to time t, is a succession of intervals in which x and t are related by a bijection x=f(t) and t = f^{-1}(x).The trajectory might be discontinuous, this would however be true.*

Consequently at any time and any place on the trajectory the system has its position related to the time by a bijection. This is the exact opposite of what the independence is. Therefore we have to replace the first postulate by a theorem, stating the bijective dependence of the position and the time. The present work will investigate the consequences of this replacement inside the classical mechanics.

Theorem of the trajectories by bijective intervals

Let us state the following theorem :

Theorem 1 : *Any trajectory of a classical parameter, recording its value with respect to time, is a succession of intervals in which the parameter and the time are related by a bijection.*

If the position is the classical parameter, the mathematical expression of this theorem in a bijective interval is the following, $\mathbf{r}$ being the vector position and t the time :

$$\mathbf{r} = f(t) \quad \text{and} \quad t = f^{-1}(\mathbf{r}) \tag{1}$$

At each single time, and each single place, on the trajectory, the mobile is necessarily in such a bijective interval, therefore it can be stated that the position is always related to the time by a bijection in classical mechanics, at any time and any place of any trajectory.

Consequences of the theorem on the Lagrangian

Because of the theorem 1, so the equation (1), the velocity in a bijective interval can be written in two different but equal ways, one only dependent upon the time, the other one only dependent upon the position :

$$\mathbf{v}(\mathbf{r}, t) = \mathbf{v}(f(t), t) = \mathbf{v}'(t) \quad \text{and} \quad \mathbf{v}(\mathbf{r}, t) = \mathbf{v}(\mathbf{r}, f^{-1}(\mathbf{r})) = \mathbf{v}''(\mathbf{r}) \tag{2}$$

Consequently we are also able to write the Lagrangian L into three other mathematical shapes, each one depending of only one single variable :

$$L = L(t, \mathbf{r}, \mathbf{v}) = A(t) = B(\mathbf{r}) = C(\mathbf{v}) \tag{3}$$

L, A, B and C are all equals, but are four different ways to express the Lagrangian as a mathematical object. We call A, B and C “bijective Lagrangians”, because they can exist only in bijective intervals.

Variation of the action

The action is defined as the integral of the Lagrangian with respect to time [1] :

$$S = \int_{t_0}^{t_1} L dt \tag{4}$$

The action is mainly used in classical mechanics through its variation, in order to get the famous Lagrange’s equations of motion. We redirect the reader to the excellent demonstration of L. Landau and E. Lifchitz explaining how to get these special equations, and we just recall them [1] :

$$\frac{\partial L}{\partial \mathbf{r}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) \quad (5)$$

In this expression the following conventions are used :

$$\frac{\partial L}{\partial \mathbf{r}} = \left(\frac{\partial L}{\partial x}, \frac{\partial L}{\partial y}, \frac{\partial L}{\partial z}\right) \text{ and } \frac{\partial L}{\partial \mathbf{v}} = \left(\frac{\partial L}{\partial v_X}, \frac{\partial L}{\partial v_Y}, \frac{\partial L}{\partial v_Z}\right) \quad (6)$$

However the theorem1 does not lead to consider the Lagrangian L, but rather the Lagrangians A, B and C. Instead of the equations (5) which are “internal” to L, we must rather look for a relationship between A, B and C. Let us then proceed in a slightly different way than the one of Lagrange.

Instead of the usual variation, we will apply a tiny variation to the action between two times, t_0 and t_1, imposing that the subsequent variation $\delta \mathbf{r}$ must be null at the limits of this interval, i.e. $\delta \mathbf{r}(t_0) = \delta \mathbf{r}(t_1) = 0$, and look at what happens to the bijective Lagrangians A, B and C. We then write :

$$\delta S = \int_{t_0}^{t_1} \frac{dA}{dt}\, \delta t\, dt = \int_{t_0}^{t_1} \frac{\partial B}{\partial \mathbf{r}}\, \delta \mathbf{r}\, dt = \int_{t_0}^{t_1} \frac{\partial C}{\partial \mathbf{v}}\, \delta \mathbf{v}\, dt = \left[\frac{\partial C}{\partial \mathbf{v}}\, \delta \mathbf{r}\right]_{t_0}^{t_1} - \int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial C}{\partial \mathbf{v}}\right) \delta \mathbf{r}\, dt \quad (7)$$

From this we see that whatever $\delta \mathbf{r}$ could be between t_0 and t_1, we shall always verify the following relationship between the bijective Lagrangians B and C :

$$\frac{\partial B}{\partial \mathbf{r}} = -\frac{d}{dt}\left(\frac{\partial C}{\partial \mathbf{v}}\right) \quad (8)$$

The former relations are the equations of motion in a bijective interval. We did not need any postulate of least action to get them. They are close to the equations (5) in their mathematical shape, but very different in their physical meaning. In addition the equations (8) are providing the new possibility to move a step forward towards the determination of the possible trajectories, what could not be done with the equations (5).

Trajectory determination

Deriving the bijective Lagrangians B and C with respect to time, and introducing the equation (8), it is trivial to see that the following relationship must be verified :

$$\frac{\partial C}{\partial \mathbf{v}}\, \mathbf{v} = \beta = \text{constant} \quad (9)$$

where β is a constant with the dimension of an energy.

The term $\partial C / \partial \mathbf{v}$ is the momentum of the system, as $\partial L / \partial \mathbf{v}$ also is, and we know that the momentum of a classical body is usually its constant mass m multiplied by its velocity : $\partial C / \partial \mathbf{v} = \mathbf{P} = m\, \mathbf{v}$. Introducing this into the relation (9), we are led to the following condition for a single motion inside a bijective interval :

$$v^2 = \frac{\beta}{m} = \text{constant} \tag{10}$$

There are only two elementary motions that can verify this very simple relationship : the uniform circular motion ($\mathbf{v} \perp \mathrm{d}\,\mathbf{v}/\mathrm{dt}$) and the uniform translation motion ($\mathrm{d}\,\mathbf{v}/\mathrm{dt} = \mathbf{0}$), which are described by the following velocities :

$$\mathbf{v_R} = \boldsymbol{\omega} \times \mathbf{r} \quad \text{with} \quad \|\mathbf{v_R}\| = v_R = \omega\, r = \text{constant} \quad \text{, and} \quad \mathbf{v_T} = \textbf{constante} \tag{11}$$

In this last definitions $\boldsymbol{\omega}$ is a rotation frequency, $\mathbf{v_R}$ is a uniform rotation velocity, and $\mathbf{v_T}$ is a uniform translation velocity.

These elementary motions are however too simple to suite all the possible bijections, $\mathbf{r} = f(t)$ and $t = f^{-1}(\mathbf{r})$, inside a bijective interval. We must then remember that the velocities are additive in classical mechanics, and the total velocity of a mobile is usually the superposition of many sub-velocities. Let us now discuss this point in more details.

Addition of velocities

We study the case where the velocity is the superposition of many sub-velocities, as described by the following equation :

$$\mathbf{v} = \sum_{i=1}^{N} \mathbf{v_i} \tag{12}$$

By integration with respect to the time, it must correspond a vector position as follows :

$$\mathbf{r} = \sum_{i=1}^{N} \mathbf{r_i} \tag{13}$$

Note that in this last expression nothing forces all the vector radius of indices i to share the same origin.

Applying the variation of the action, as described above, on the bijective Lagrangian B and C, depending of such a composed vector radius for the first, and of such a composed velocity for the second, leads to the equivalent of the relation (8) :

$$\sum_{i=1}^{N} \left[\frac{\partial B}{\partial \mathbf{r_i}} + \frac{d}{dt}\left(\frac{\partial C}{\partial \mathbf{v_i}} \right) \right] \delta\, \mathbf{r_i} = \mathbf{0} \tag{14}$$

This relationship must be true whatever the values of the different $\delta\, \mathbf{r_i}$ can be. An obvious solution for this equation then occurs if each sub-motion respects the relation (8) individually, so :

$$\frac{\partial B}{\partial \mathbf{r_i}} + \frac{d}{dt}\left(\frac{\partial C}{\partial \mathbf{v_i}} \right) = \mathbf{0} \qquad \forall\ i \tag{15}$$

Now deriving B and C with respect to time, and introducing the relationship (15), we obtain the equivalent of the equation (9) that we found above for a unique motion :

$$\sum_{i=1}^{N} \frac{\partial C}{\partial \mathbf{v_i}} \mathbf{v_i} = \text{constant} \tag{16}$$

Considering that $\partial C / \partial \mathbf{v_i} = \mathbf{P_i} = m\,\mathbf{v_i}$ is the partial momentum due to the motion i, m being the mass of the body, and $\mathbf{v_i}$ its partial velocity, the system must verify the following condition :

$$\sum_{i=1}^{N} v_i^2 = \text{constant} \tag{17}$$

This relationship is always true as far as each partial velocity respects the conditions (11), i.e. if the total velocity is the superposition of one or many uniform translation and rotation velocities. Therefore we can write the total velocity as so :

$$\mathbf{v} = \sum_{i=1}^{N} \mathbf{v_{R\,i}} + \sum_{j=1}^{M} \mathbf{v_{T\,j}} = \mathbf{v_R} + \mathbf{v_T} \quad \text{where N and M are integers} \tag{18}$$

In this expression each $\mathbf{v_{R\,i}}$ and each $\mathbf{v_{T\,j}}$ is respecting the relations (11).

This last equation can produce any bijective trajectory, $\mathbf{r} = f(t)$ and $t = f^{-1}(\mathbf{r})$, suitable inside a bijective interval. Remember however that this only applies to mobiles which partial momentum is equal to their mass multiplied by their partial velocity.

One of the most simple velocity superposition is the addition of a single uniform rotation velocity with a single uniform translation velocity, that leads to the pure Keplerian motion, as stated by the theorem of the Keplerian kinematics [2].

A very important remark has to be done here : the relation (18) being fundamentally periodic it is trivial that it is nothing else but a Fourier series. To make it obvious, let place all the $\mathbf{v_{R\,i}}$ and $\mathbf{v_{T\,j}}$ in the same geometric plane, and look at each X and Y dimensions, they will exhibit each a pure Fourier series.

Generalization to all physical parameters

Fundamental equations

The variation of the action presented here concerned the trajectory of a mobile in three dimensions. However as far as we use no postulate, there is no reason why the present approach could not be used for any other physical quantity evolving in time.

When studying a single physical quantity, let us call it q, instead of a three dimension position, it is important to note that nothing changes from what has been exposed here, except that we can integrate the equation (9) with respect to time in order to get :

$$\Delta C = C - C_0 = \beta \ln \left(\frac{\dot{q}}{\dot{q}_0} \right) \quad \text{or} \quad \dot{q} = \dot{q}_0 \, e^{\frac{\Delta C}{\beta}} \tag{19}$$

where β is a constant. In this expression the Lagrangian C is dedicated to the study of the quantity $\dot{q}$, but not any more to the kinematics velocity. Therefore the notion of momentum $P = \partial C / \partial \dot{q}$, has to be taken in a wide sens here, because in general it does not refer to the kinematics momentum. The same notice is also relevant for the force, which is the derivative of the momentum with respect to time, $F = d\,P/dt$. This being said we can defined the “generalized” momentum and force as so :

$$P = \frac{\partial C}{\partial \dot{q}} = \frac{\beta}{\dot{q}} \quad \text{and} \quad F = \frac{d}{dt} \left(\frac{\partial C}{\partial \dot{q}} \right) = - \frac{\beta}{\dot{q}^2} \ddot{q} \tag{20}$$

Entropy

The above formulations are concerning a unique velocity $\dot{q}$, but in general the real velocity is the addition of many sub-velocities. However, each of these lasts has to verify the relation (19), so we can write :

$$\dot{q} = \sum_{n=0}^{N} \dot{q}_n = \sum_{n=0}^{N} \dot{q}_n^0 \, e^{\frac{\Delta C_n}{\beta_n}} \quad \text{where the} \quad \dot{q}_n^0 \quad \text{are constant} \tag{21}$$

If we divide the above expression by a constant reference speed $\dot{q}_0$, we then describe a statistical partition function [3] :

$$Z = \sum_{n=0}^{N} \frac{\dot{q}_n^0}{\dot{q}_0} \, e^{\frac{\Delta C_n}{\beta_n}} \tag{22}$$

Replacing this result into the equation (19), we get :

$$\Delta C = \beta \ln(Z) \tag{23}$$

Now, if ΔC is the Gibbs free energy [3], and $\beta = kT$ is the product of the Boltzmann’s constant k by the temperature T, the former expression is the Boltzmann’s formula of the statistical entropy :

$$S_{ent} = k \ln(Z) \tag{24}$$

Chemical kinetics and ideal gas law

It must also be noted that $\dot{q}$ can be expressed as a function of the single variable t, because of the theorem 1. Because of the relation (19), the most simple way of respecting this obligation is to have a Lagrangian simply proportional to the time :

$$\Delta C = \beta \, \nu \, \Delta t \quad \text{with} \quad \Delta t = t - t_0 \tag{25}$$

In this condition the quantity, its generalized speed and acceleration must respect the following equations :

$$q = q_0 \, e^{\nu \Delta t} \quad , \quad \dot{q} = \nu q \quad \text{and} \quad \ddot{q} = \nu^2 q \tag{26}$$

These last expressions are typical of the chemical kinetics, or the radioactive decay, when measuring the quantity of a reactive.

In addition, replacing the relations (26) into the relations (20), we see that the following relation exists :

$$F \, q = -\beta \tag{27}$$

If q is the volume of a gas, F can only be the pressure, and the relation (27) is nothing else but the ideal gas law, at the condition that $\beta = N \, kT$, N being the number of molecules, k the Boltzmann’s constant and T the temperature.

Discussion

Somehow the validity of the first postulate of the classical mechanics, pretending the independence of the position and the time, could have been questioned before. If we make a short simplification, it states that for a given set of physical properties, at a given time a system **could** occupy different positions. Indeed at noon you **could** be either at home or in your office, but at noon a Keplerian orbiter **can** only be at a single and specific position, deeply depending upon the time. Since ever the ephemeris described such a bijection relating the position of the astral bodies and the time. Looking at the position of the astral bodies was even the first mean for the humanity to determine what date and time it was, with no doubt.

The experiment then forces us to replace the first postulate by the theorem 1, and to consider that at any time and any place on the trajectory the position and the time are related by a bijection. This is not only true for the position in three dimensions but also for any physical parameter measured with respect to time.

Such a replacement has of course a deep impact on the search for the equations of motion, i.e. on the way to run the variation of the action. Nonetheless the constraints imposed by the bijection, lead to a simplification, and to final equations rather close to those of Lagrange. Incidentally to achieve so we had no need of the least action principle, that becomes therefore useless.

These new equations of motion have an advantage on their Lagrange’s equivalent, they give more information on the possible trajectories. So for a classical system having a momentum equal to its mass multiplied by its velocity, only two elementary motions are allowed : the uniform rotation and the uniform translation. Furthermore the velocities being additive in classical mechanics, any velocity will then be the addition of one or many of these two elementary uniform velocities, of rotation and translation.

This special structure of the velocity, composed of elementary periodic sub-velocities, is nothing else but the one of a Fourier series. So we can see that the big bang of replacing the first postulate by an opposite theorem, leads finally to a very reasonable result : all classical velocities can be decomposed in a Fourier series, which terms are each representing an elementary sub-velocity. We talk here of the velocity in a wide sens, i.e. the derivative of any physical parameter with respect to the time. No one will be surprised by this revelation, as we all experienced the decomposition of a periodic curve into a Fourier series, in nearly all the fields of science. In this sens the Fourier series does not appear any more as only a fantastic mathematical tool, but above all as the deep track of the structure of the motion in our universe.

Also note that one of the the most simple addition of velocities that we can imagine is the one of a single uniform rotation velocity with a single uniform translation velocity. The result is the velocity of a Keplerian orbiter, as stated by the theorem of the Keplerian kinematics. Therefore we know now the reason why the nature chose to exhibit the Kepler's laws, instead of any other imaginable motion : the constraint of the bijective dependence of the position and the time, let no other possibility.

In the case where we study a single physical parameter, we must keep the present approach, but in one dimension, and we are then able to determine the mathematical shape of the Lagrangian. Here again no big bang occurs for this Lagrangian as it is consistent with the well known logarithmic structure encountered in thermodynamics. As an illustration we show that it gives straight forward the Boltzmann's definition of the statistical entropy, the ideal gas law and the equations leading the chemical kinetics. However these thermodynamics issues deserves a complete study to be fully investigated from the present point of view.

In the present work we used no postulate, nor hypothesis, nor unprovable theory, but removed two of them : the time-position independence, and the least action principle. Although we had to change the usual calculus of variation, we only simplified it, once again by admitting no new postulate. The result is not revolutionary, as it forecasts what is already observed experimentally (Fourier series, Keplerian motion, Boltzmann's entropy, …), but it gives a better overall understanding of the motion in classical mechanics, unifying domains that were far from being related before, as the Keplerian motion and the Boltzmann's entropy for instance. We hope then to enforce the basement of the classical mechanics with the present contribution.

A conclusion that we can get from the present work is to take care about postulates. An ideal theory of physics should be free of any postulate. Unfortunately some time we need to use them, because if we accept this imperfection, we nonetheless get an overall benefit. So let us do this way, but let us then always remember that there is some weak points in the theory, that make it fragile. Whoever the genius stating a postulate, we shall always warn the students that his theory has a weak point, and therefore has to be improved.

References:

[1] L.Landau & E. Lifchitz, "Mechanics", Ed Mir Moscou 1966

[2] H. Le Cornec, Theorem of the Keplerian Kinematics, http://vixra.org/abs/1504.0128

[3] L.Landau & E. Lifchitz, "Statistical Mechanics", Ed Mir Moscou 1984