Refutation of non Sahlqvist formulas by three counter examples

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Abstract: We evaluate three equations as examples of non Sahlqvist formulas. None is tautologous. What follows is that Fine’s theorem and monotonic modal logic are refuted. Therefore those conjectures form a non tautologous fragment of the universal logic $VŁ4$.

We assume the method and apparatus of Meth8/$VŁ4$ with $\text{Tautology}$ as the designated proof value, $\mathbf{F}$ as contradiction, $\mathbf{N}$ as truthity (non-contingency), and $\mathbf{C}$ as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET $\sim$ Not, $\neg$ ; $+$ Or, $\lor$, $\cup$ ; $-$ Not Or; $\&$ And, $\land$, $\cap$ ; $\setminus$ Not And;
$>$ Imply, greater than, $\rightarrow$, $\Rightarrow$, $\supset$ ; $<$ Not Imply, less than, $\in$, $\subset$, $\varsubsetneq$, $\varsupsetneq$, $\approx$;
$=$ Equivalent, $\equiv$, $\iff$; $\neq$ Not Equivalent, $\not\equiv$;
$\%$ possibility, for one or some, $\exists$, $\emptyset$, $\Diamond$; $\#$ necessity, for every or all, $\forall$, $\Box$, $\square$;
$(z=z)$ $\top$ as tautology, $\top$, ordinal 3; $(z@z)$ $\bot$ as contradiction, $\bot$, Null, $\perp$, zero;
$(\%z>\#z)$ $\Delta$ as non-contingency, $\Delta$, ordinal 1; $(\%z<\#z)$ $\nabla$ as contingency, $\nabla$, ordinal 2;
$\neg(y<x)$ $(x\leq y)$, $(x\subseteq y)$; $(A=B)$ $(A\sim B)$; $(B>A)$ $(A\neg B)$; $(B>A)$ $(A\neg B)$.
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: en.wikipedia.org/wiki/Sahlqvist_formula

Examples of three non-Sahlqvist formulas:

1. The McKinsey formula does not have a first-order frame condition.

   $$\Box \Diamond p \rightarrow \Diamond \Box p$$  \hspace{1cm} (1.1)

   $$\#(\%p>\#p) ; \hspace{2cm} \text{NNNN} \hspace{0.5cm} \text{NNNN} \hspace{0.5cm} \text{NNNN} \hspace{0.5cm} \text{NNNN}$$  \hspace{1cm} (1.2)

2. The Löb axiom does not have a first-order frame condition.

   $$\Box(\Box p \rightarrow p) \rightarrow \Box p$$  \hspace{1cm} (2.1)

   $$\#(\#p>\#p) ; \hspace{2cm} \text{CTCT} \hspace{0.5cm} \text{CTCT} \hspace{0.5cm} \text{CTCT} \hspace{0.5cm} \text{CTCT}$$  \hspace{1cm} (2.2)

3. The conjunction of the McKinsey formula and the [modal] (4) axiom has a first-order frame condition ... but is not equivalent to any Sahlqvist formula.

   $$\Box(\Box p \rightarrow \Box p) \land (\Diamond \Diamond q \rightarrow \Diamond q)$$  \hspace{1cm} (3.1)

   $$\#(\#p>\#p) \& (\%q>\%q) ; \hspace{2cm} \text{NNNN} \hspace{0.5cm} \text{NNNN} \hspace{0.5cm} \text{NNNN} \hspace{0.5cm} \text{NNNN}$$  \hspace{1cm} (3.2)

Eqs. 1.2-3.2 are not tautologous and refute the conjecture of non Sahlqvist formulas as tautologous. What follows is that Fine’s theorem and monotonic modal logic are also refuted.