A short proof that quantum entropy equals space-time curvature

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We introduce the statistical physics of space-time events. We call the framework: geometric thermodynamics. As the main result we show that for all systems of geometric thermodynamics, the quantum entropy equals the space-time entropy equals the curvature of space-time.

1 Introduction

What can be more well-suited to a statistical treatment than a set of randomly selected events?

In this paper, we will introduce the statistical physics of space-time events. Two key factors will be at play:

- Since events are defined using orthogonal coordinates, then the quantities used in the partition function of the statistical ensemble will be non-commutative operators. Consequently, the partition function will be a full fledged quantum partition function with all the features normally attributed to quantum systems: including operators, Von Neumann entropy, measurement basis, interference terms, etc.
- Since the statistical ensemble is defined over a set of general space-time events, the structure of space-time including the curvature in the form a general metric is recovered as a thermodynamic equation of state.

As the main result, we show that the entropy of this system, quantum in nature, is equated to the curvature of space-time. The curvature is thus emergent as an equivalent representation of the entropy of the quantum system.

This paper assumes that the reader is as least slightly familiar with Clifford algebras, relativity and fully familiar with statistical physics.

1.1 Recap: statistical physics

In statistical physics, we are interested in the distribution that maximizes the Boltzmann entropy,

$$S = -k_B \sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) \tag{1}$$

subject to the fixed macroscopic quantities (the statistical priors). The solution for this is the Gibbs ensemble. Typical thermodynamic quantities are shown in Table 1.

Symbol	Name	Units	Type
$E(q)$	energy	[Joule]	extensive
$1/T = k_B\beta$	temperature	1/[Kelvin]	intensive
$\overline{E}$	average energy	[Joule]	macroscopic
$V(q)$	volume	[meter^3]	extensive
$p/T = k_B\gamma$	pressure	[Joule/(Kelvin-meter^3)]	intensive
$\overline{V}$	average volume	[meter^3]	macroscopic
$N(q)$	number of particles	[kg]	extensive
$-\mu/T = k_B\delta$	chemical potential	[Joule/(Kelvin-kg)]	intensive
$\overline{N}$	average number of particles	[kg]	macroscopic

Table 1: Typical thermodynamic quantities

Taking these quantities as examples, the partition function (Gibbs ensemble) becomes:

$$Z = \sum_{q \in \mathbb{Q}} e^{-\beta E(q) - \gamma V(q) - \delta N(q)} \tag{2}$$

The probability of occupation of a micro-state (Gibbs measure) is:

$$\rho(q) = \frac{1}{Z} e^{-\beta E(q) - \gamma V(q) - \delta N(q)} \tag{3}$$

The average values and their variance are for each quantity:

$$\overline{E} = \sum_{q \in Q} \rho(q) E(q) \qquad \overline{E} = \frac{-\partial \ln Z}{\partial \beta} \qquad \overline{(\Delta E)^2} = \frac{\partial^2 \ln Z}{\partial \beta^2} \tag{4}$$

$$\overline{V} = \sum_{q \in Q} \rho(q) V(q) \qquad \overline{V} = \frac{-\partial \ln Z}{\partial \gamma} \qquad \overline{(\Delta V)^2} = \frac{\partial^2 \ln Z}{\partial \gamma^2} \tag{5}$$

$$\overline{N} = \sum_{q \in Q} \rho(q) N(q) \qquad \overline{N} = \frac{-\partial \ln Z}{\partial \delta} \qquad \overline{(\Delta N)^2} = \frac{\partial^2 \ln Z}{\partial \delta^2} \tag{6}$$

The entropy can be obtained from the partition function and is given by:

$$S = k_B \left(\ln Z + \beta \overline{E} + \gamma \overline{V} + \delta \overline{N} \right) \tag{7}$$

The laws of thermodynamics can be recovered by taking the following derivatives:

$$\left.\frac{\partial S}{\partial \overline{E}}\right|_{\overline{V},\overline{N}} = \frac{1}{T} \qquad \left.\frac{\partial S}{\partial \overline{V}}\right|_{\overline{E},\overline{N}} = \frac{p}{T} \qquad \left.\frac{\partial S}{\partial \overline{N}}\right|_{\overline{E},\overline{V}} = -\frac{\mu}{T} \tag{8}$$

and grouping them as follows:

$$d\overline{E} = TdS - pd\overline{V} + \mu d\overline{N} \tag{9}$$

This is the equation of state of the system.

1.2 *Entropy and space-time*

An equivalence relation between quantum entropy and space-time has been (or the very least could have been) anticipated since probably the better part of four decades. The first hints were provided by the work of Jacob Bekenstein[3] regarding the similarities between black holes and thermodynamics, culminating in the four laws of black hole thermodynamics. The temperature, originally introduced by analogy, was soon augmented to a real notion by Steven Hawking with the discovery of the Hawking temperature derivable from quantum field theory on curved space-time[4]. Another very significant result is that of the Bekenstein-Hawking entropy[5] connecting the area of the surface of a horizon to be proportional to one fourth the number of elements with Planck area that can be fitted on the surface: $S = k_B c^3/(4\hbar G)A$. The hint strongly suggests that quantum theory meets general relativity at the surface of black holes.

[3] Jacob D Bekenstein. Generalized second law of thermodynamics in black-hole physics. *Physical Review D*, 9(12):3292, 1974; and Jacob D Bekenstein. Generalized second law of thermodynamics in black-hole physics. *Physical Review D*, 9(12):3292, 1974

[4] Stephen W Hawking. Black hole explosions? *Nature*, 248(5443):30, 1974

[5] Jacob D Bekenstein. Black-hole thermodynamics. *Physics Today*, 33(1): 24–31, 1980

Yet another key results is that of Ted Jacobson[6] who directly recovered the Einstein field equation as an equation of state of a suitable thermodynamic system. To justify the emergence of general relativity from entropy, Jacobson first postulated that the energy flowing out of horizons becomes hidden from observers. Then he attributed the role of heat to this energy for the same reason that heat is energy that is inaccessible for work. In this case, its effects are felt, not as "warmth", but as gravity originating from the horizon. Finally, with the assumption that the heat is proportional to the area A of the system under some proportionality constant η, and some legwork, the Einstein field equations are eventually recovered. This is the essence of the argument presented by Ted Jacobson.

[6] Ted Jacobson. Thermodynamics of spacetime: The einstein equation of state. *Phys. Rev. Lett.*, 75:1260–1263, Aug 1995. DOI: 10.1103/PhysRevLett.75.1260. URL https://link.aps.org/doi/10.1103/PhysRevLett.75.1260

In 2011, Erik Verlinde[7] proposed an entropic derivation of the classical law of inertia and of classical gravity. He compared the emergence of such laws to that of an entropic force such as a polymer in a warm bath. Each law is emergent from the equation $TdS = Fdx$, under the appropriate temperature and a posited entropy relation.

[7] Erik P. Verlinde. On the origin of gravity and the laws of newton. *Journal of High Energy Physics*, 2011(4):29, Apr 2011. ISSN 1029-8479. DOI: 10.1007/JHEP04(2011)029. URL https://doi.org/10.1007/JHEP04(2011)029

His proposal has encouraged a plurality of attempts to reformulate known laws of physics using the framework of statistical physics. Matt Visser (in Visser [2011]) provides, in the introduction to his paper, a good summary of the literature on the subject. The ideas of Verlinde have been applied to loop quantum gravity (Smolin [2010]), the Coulomb force (Wang [2010]), Yang-Mills gauge fields (Freund [2010]) and cosmology (Cai et al. [2010], Li and Wang [2010], Easson et al. [2011]). Some criticism has however been voiced, notably by (Hossenfelder [2010], Kobakhidze [2011a], Gao [2011], Hu [2011], Kobakhidze [2011b]), and by Visser himself in (Visser [2011]).

Finally and even more recently, a connection between entanglement and general relativity as been supported by multiple authors: Swingle [2012], Van Raamsdonk [2010], Evenbly and Vidal [2011], Faulkner et al. [2014, 2017], Czech et al. [2016], Maldacena [1999], Swingle and Van Raamsdonk [2014], Maldacena and Susskind [2013], Sanches and Weinberg [2016], Susskind [2016], Cao et al. [2017], Bao et al. [2017b,a], Susskind [2017], Padmanabhan [2010], Jacobson [2016], Carroll and Remmen [2016].

Here, we derive an equivalence relation between quantum entropy in general (not just entanglement) and the curvature of space-time.

2 Geometric thermodynamics

Definition 1 (Events). *We define a set of events* $\mathbb{Q}$ *as:*

$$\mathbb{Q} \subset \{q : q \in R^{n+m}\} \tag{10}$$

The quantity $n + m$ *denotes the number of dimensions of the events. By convention we pose that* n *denotes the number of time dimensions and* m *denotes the number of space dimensions. An event* q *is represented by an* $(n + m)$*-tuple:*

$$q := (\underbrace{X_0, ...X_{n-1}}_{\text{time terms}}, \underbrace{X_n, ...X_{n+m-1}}_{\text{space terms}}) \tag{11}$$

where each member of the tuple is an element of $\mathbb{R}$*.*

Definition 2 (Algebra of events). *In order of simplest to more general, we define an algebraic basis for events as follows:*

- *For Euclidean space, we use the basis* $\sigma_x, \sigma_y, \sigma_z$ *with the following properties:*

$$\sigma_x^2 = 1 \tag{12}$$
$$\sigma_y^2 = 1 \tag{13}$$
$$\sigma_z^2 = 1 \tag{14}$$
$$\sigma_x\sigma_y + \sigma_y\sigma_x = 0 \tag{15}$$
$$\sigma_x\sigma_z + \sigma_z\sigma_x = 0 \tag{16}$$
$$\sigma_y\sigma_z + \sigma_z\sigma_y = 0 \tag{17}$$

  One instance of the elements $\sigma_x, \sigma_y, \sigma_z$ *are simply the well known Pauli matrices. These elements corresponds to a basis of the Clifford algebra* $Cl(0, 3)$*.*

- *For Lorentzian space-time (3+1), we use the basis* $\gamma_x, \gamma_y, \gamma_z, \gamma_t$ *with the following properties:*

$$\gamma_x^2 = -1 \tag{18}$$
$$\gamma_y^2 = -1 \tag{19}$$
$$\gamma_z^2 = -1 \tag{20}$$
$$\gamma_t^2 = 1 \tag{21}$$
$$\gamma_x\gamma_y + \gamma_y\gamma_x = 0 \tag{22}$$
$$\gamma_x\gamma_z + \gamma_z\gamma_x = 0 \tag{23}$$
$$\gamma_x\gamma_t + \gamma_t\gamma_x = 0 \tag{24}$$
$$\gamma_y\gamma_z + \gamma_x\gamma_y = 0 \tag{25}$$
$$\gamma_y\gamma_t + \gamma_t\gamma_y = 0 \tag{26}$$
$$\gamma_z\gamma_t + \gamma_t\gamma_z = 0 \tag{27}$$

One instance of the elements $\gamma_x, \gamma_y, \gamma_z, \gamma_t$ are simply the well known Dirac matrices. These elements corresponds to a basis of the Clifford algebra $Cl(1,3)$.

- *For a general ortho-normal basis, we use a Clifford algebra $Cl(n,m)$. First, one defines a diagonal metric $\eta_{\mu\nu}$ with a signature. For instance, in $3+1$ space-time with signature $+---$, η is:*

$$\eta = \begin{array}{c} \begin{matrix} \mu=0 & \mu=1 & \mu=2 & \mu=3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{matrix} \nu=0 \\ \nu=1 \\ \nu=2 \\ \nu=3 \end{matrix} \end{array} \tag{28}$$

Then $\forall \mu \in \{0, ..., m+n-1\} \wedge \forall \nu \in \{0, .., m+n-1\}$ the properties of the basis elements are:

$$\frac{1}{2}(e_\mu e_\nu + e_\nu e_\mu) = \eta_{\mu\nu} \tag{29}$$

- *Finally, for a general (not necessarily ortho-normal) basis, the properties of the basis elements of the algebra are:*

$$\frac{1}{2}(e_\mu e_\nu + e_\nu e_\mu) = g_{\mu\nu} \tag{30}$$

where

$$g = \begin{array}{c} \begin{matrix} \mu=0 & \mu=1 & \mu=2 & \mu=3 \end{matrix} \\ \begin{pmatrix} g_{00} & g_{10} & g_{20} & g_{30} \\ g_{01} & g_{11} & g_{21} & g_{31} \\ g_{02} & g_{12} & g_{22} & g_{32} \\ g_{03} & g_{13} & g_{23} & g_{33} \end{pmatrix} \begin{matrix} \nu=0 \\ \nu=1 \\ \nu=2 \\ \nu=3 \end{matrix} \end{array} \tag{31}$$

Then using such a basis, we write the $(m+n)$-tuple of an event q as an algebraic equation:

$$\vec{q} := \underbrace{X_0e_0 + ... + X_{n-1}e_{n-1}}_{\text{time terms}} + \underbrace{X_ne_n + ... + X_{n+m-1}e_{n+m-1}}_{\text{space terms}} \tag{32}$$

As an example; in the case of the Lorentz space-time, we would write an event $q = (t, x, y, z)$ algebraically as:

$$\vec{q} := \underbrace{t\gamma_t}_{\text{time term}} + \underbrace{x\gamma_x + y\gamma_y + z\gamma_z}_{\text{space terms}} \tag{33}$$

Definition 3 (Interval). *Using the basis for $Cl(1,3)$ (a.k.a Dirac algebra), let*

$$\vec{q}_1 := t_1\gamma_t + x_1\gamma_x + y_1\gamma_y + z_1\gamma_z \tag{34}$$
$$\vec{q}_2 := t_2\gamma_t + x_2\gamma_x + y_2\gamma_y + z_2\gamma_z \tag{35}$$

Then, the interval between these events is:

$$(\vec{q}_1 - \vec{q}_2)^2 = \big((t_1-t_2)\gamma_t + (x_1-x_2)\gamma_x + (y_1-y_2)\gamma_y + (z_1-z_2)\gamma_z\big)^2 \tag{36}$$
$$= \big((\Delta t)\gamma_t + (\Delta x)\gamma_x + (\Delta y)\gamma_y + (\Delta z)\gamma_z\big)^2 \tag{37}$$
$$= (\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \tag{38}$$

Likewise, using a general (not necessarily ortho-normal) basis, the vectors are:

$$\vec{q}_1 := t_1e_t + x_1e_x + y_1e_y + z_1e_z \tag{39}$$
$$\vec{q}_2 := t_2e_t + x_2e_x + y_2e_y + z_2e_z \tag{40}$$

and the interval is defined using the familiar metric tensor g:

$$(\vec{q}_1 - \vec{q}_2)^2 = \big((t_1-t_2)e_t + (x_1-x_2)e_x + (y_1-y_2)e_y + (z_1-z_2)e_z\big)^2 \tag{41}$$
$$= \big((\Delta t)e_t + (\Delta x)e_x + (\Delta y)e_y + (\Delta z)e_z\big)^2 \tag{42}$$
$$\begin{aligned}=&(\Delta t)e_t(\Delta t)e_t + (\Delta t)e_t(\Delta x)e_x + (\Delta t)e_t(\Delta y)e_y + (\Delta t)e_t(\Delta z)e_z\\ &+ (\Delta x)e_x(\Delta t)e_t + (\Delta x)e_x(\Delta x)e_x + (\Delta x)e_x(\Delta y)e_y + (\Delta x)e_x(\Delta z)e_z\\ &+ (\Delta y)e_y(\Delta t)e_t + (\Delta y)e_y(\Delta x)e_x + (\Delta y)e_y(\Delta y)e_y + (\Delta y)e_y(\Delta z)e_z\\ &+ (\Delta z)e_z(\Delta t)e_t + (\Delta z)e_z(\Delta x)e_x + (\Delta z)e_z(\Delta y)e_y + (\Delta z)e_z(\Delta z)e_z\end{aligned} \tag{43}$$
$$\begin{aligned}=&(\Delta t)^2e_t^2 + (\Delta x)^2e_x^2 + (\Delta y)^2e_y^2 + (\Delta z)^2e_z^2\\ &+ \Delta t\Delta x(e_te_x + e_te_x) + \Delta t\Delta y(e_te_y + e_ye_t) + \Delta t\Delta z(e_te_z + e_ze_t)\\ &+ \Delta x\Delta y(e_xe_y + e_ye_x) + \Delta x\Delta z(e_xe_z + e_ze_x)\\ &+ \Delta y\Delta z(e_ye_z + e_ze_y)\end{aligned} \tag{44}$$
$$= \sum_{\mu\nu} g_{\mu\nu}\Delta X_\mu \Delta X_\nu \tag{45}$$

where $g_{\mu\nu} = \frac{1}{2}(e_\mu e_\nu + e_\nu e_\mu)$.

Definition 4 (Relevant physical quantities)**.** *We first introduce two physical quantities: 1) the entropic repetency $\tilde{k}$ (normally this quantity is represented by the symbol ν but we already use this symbol extensively for the indices of our basis therefore we opt to use $\tilde{k}$ here) and 2) the entropic frequency f. Specifically, $\tilde{k} = k/2\pi = 1/\lambda$, where k is the wave-number and λ is the wavelength.*

These quantities are the conjugated variables to a distance x and a time t, respectively. By convention, we prefix the Lagrange multipliers with the word "entropic", and its averaged conjugated quantity will be prefixed with the word "thermal". $\tilde{k}$ and f are both intensive properties, whereas x and t are extensive. Indeed, a process taking 1 minute followed by a process taking 2 minutes takes a total of 3 minutes (extensive). For the x quantity; walking 1 meter followed by walking 2 meters implies one has walked a total of 3 meters (extensive). Adding or removing clocks from a group of clocks ticking at a frequency f (say once per second) has no impact on the frequency of the other elements of the group (intensive). The same argument applies to the entropic repetency (intensive). The units of $\tilde{k}$ are m^{-1}, the units of x are m, the units of t are s and the units of f are s^{-1}. Finally, we define the quantity $c := f/\tilde{k}$. These quantities are summarized in Table 2.

Symbol	**Name**	**Units**	**Type**
$x(q)$	space	[meter]	extensive
$\tilde{k}$	entropic repetency	[1/meter]	intensive
$\overline{x}$	thermal space	[meter]	macroscopic
$t(q)$	time	[second]	extensive
f	entropic frequency	[1/second]	intensive
$\overline{t}$	thermal time	[second]	macroscopic
$c := f/\tilde{k}$	entropic speed	[meter/second]	intensive

Table 2: Relevant physical quantities of the ensemble

Definition 5 (Ensemble of events)**.** *An ensemble of events $\mathbb{Q}$ is the probability distribution $\rho(q)$ which maximizes the Shannon entropy (in base e):*

$$S = -\sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) \tag{46}$$

using the macroscopic priors defined in table 2 for events expressed in the algebra of events $Cl(n, m)$. Specifically, $\forall i \in \{0, ..., n+m-1\}$ the priors to the ensemble are:

$$\langle X_i e_i \rangle = \sum_{q \in \mathbb{Q}} X_i[q] e_i \rho(q) \tag{47}$$

The functions $X_i[q]$ are maps $X_i : \mathbb{Q} \to \mathbb{R}$ where $X_i[q]$ returns the value of the i^{th} element of the $(m+n)$-tuple of q.

Theorem 1 (Partition function)**.** *The partition function of the ensemble of events is:*

$$Z = \sum_{q \in Q} \exp\left(-f \underbrace{(X_0[q]e_0 + ... + X_{n-1}[q]e_{n-1})}_{\text{time terms}} - \tilde{k} \underbrace{(X_n[q]e_n + ... + X_{n+m-1}[q]e_{n+m-1})}_{\text{space terms}} \right) \tag{48}$$

where $\tilde{k}$ and f are Lagrange multipliers. The probability distribution $\rho(q)$ is:

$$\rho(q) = \frac{1}{Z} \exp\left(-f \underbrace{(X_0[q]e_0 + ... + X_{n-1}[q]e_{n-1})}_{\text{time terms}} - \tilde{k} \underbrace{(X_n[q]e_n + ... + X_{n+m-1}[q]e_{n+m-1})}_{\text{space terms}} \right) \tag{49}$$

Proof. One obtains the partition function Z with the usual method of the Lagrange multipliers.

1. The constraints are:

$$1 = \sum_{q \in Q} \rho(q) \tag{50}$$

$$\langle X_i e_i \rangle = \sum_{q \in Q} X_i[q] e_i \rho(q) \tag{51}$$

2. The Lagrange equation is:

$$\mathcal{L} = \left(-\sum_{q \in Q} \rho(q) \ln \rho(q) \right) - \lambda \left(\sum_{q \in Q} \rho(q) - 1 \right) - \sum_i \left(\lambda_i \sum_{q \in Q} X_i[q] e_i \rho(q) - \langle X_i e_i \rangle \right) \tag{52}$$

   where λ and the set of λ_i are Lagrange multipliers.

3. Maximizing $\mathcal{L}$ with respect to $\rho(q)$ is done by taking its derivative and posing it equal to zero:

$$\frac{\partial \mathcal{L}}{\partial \rho(q)} = -\ln \rho(q) - 1 - \lambda - \sum_i (\lambda_i X_i[q] e_i) = 0 \tag{53}$$

4. Solving for $\rho(q)$ one obtains:

$$\rho(q) = \exp(-1 - \lambda) \exp\left(-\sum_i \lambda_i X_i[q] e_i \right) \tag{54}$$

5. From the constraint $1 = \sum_{q \in Q} \rho(q)$, we can find the expression for $\exp(-1 - \lambda)$ as follows:

$$1 = \sum_{q \in \mathbb{Q}} \rho(q) \tag{55}$$

$$= \sum_{q \in \mathbb{Q}} \exp(-1-\lambda) \exp\left(-\sum_i \lambda_i X_i[q] e_i\right) \tag{56}$$

$$\implies \exp(-1-\lambda) = \frac{1}{\sum_{q \in \mathbb{Q}} \exp\left(-\sum_i \lambda_i X_i[q] e_i\right)} \tag{57}$$

6. We then define the inverse of the last term as the partition function:

$$Z := \sum_{q \in \mathbb{Q}} \exp\left(-\sum_i \lambda_i X_i[q] e_i\right) \tag{58}$$

7. Finally, we write $\rho(q)$ using Z. We obtain:

$$\rho(q) = \frac{1}{Z} \exp\left(-\sum_i \lambda_i X_i[q] e_i\right) \tag{59}$$

□

Theorem 2 (Geometric entropy)**.** *The entropy of the ensemble of events is:*

$$\hat{S} = \ln z + \underbrace{f\langle X_0 e_0\rangle + \ldots + f\langle X_{n-1} e_{n-1}\rangle}_{\text{time terms}} + \underbrace{\tilde{k}\langle X_n e_n\rangle + \ldots + \tilde{k}\langle X_{n+m-1} e_{n+m-1}\rangle}_{\text{space terms}} \tag{60}$$

We interpret the entropy as the information one gains by knowing which event q was randomly selected from the set of events $\mathbb{Q}$ under the probability distribution $\rho(q)$.

- *Note: the geometric entropy is not an element of the reals, but a multivector of the Clifford algebra. This situation is undesirable —one would prefer a definition of entropy that is a real number— and in fact it will be by resolving this problem that we will obtain the interesting physics connecting the quantum entropy to the curvature of space-time. In the next section we will explore how to express this entropy more attractively as an element of $\mathbb{R}_{\geq 0}$.*

Proof. The Shannon entropy in base e is:

$$S = -\sum_{q \in \mathbb{Q}} \rho(q) \ln \rho(q) \tag{61}$$

Replacing $\rho(q)$ in Equation 61 with $\rho(q)$ in Equation 49, one obtains:

$$S = -\sum_{q\in Q} \frac{1}{Z} \exp\left(-fX_0[q]e_0 + ...\right) \ln \frac{1}{Z} \exp\left(-fX_0[q]e_0 + ...\right) \tag{62}$$

The terms in the exponential beyond the first one have been omitted for brevity. With a few rearrangements, one obtains:

$$S = -\frac{1}{Z} \sum_{q\in Q} \exp\left(-fX_0[q]e_0 + ...\right) \left[\ln \exp\left(-fX_0[q]e_0 + ...\right) - \ln Z\right] \tag{63}$$

The logarithm of the exponential of a matrix is equal to the matrix $\ln \exp A = A$. Therefore,

$$S = -\frac{1}{Z} \sum_{q\in Q} \exp\left(-fX_0[q]e_0 + ...\right) \left[\left(-fX_0[q]e_0 + ...\right) - \ln Z\right] \tag{64}$$

$$= -\frac{1}{Z} \sum_{q\in Q} \left(-fX_0[q]e_0 + ...\right) \exp\left(-fX_0[q]e_0 + ...\right) + \frac{1}{Z} \sum_{q\in Q} \exp\left(-fX_0[q]e_0 + ...\right) \ln Z \tag{65}$$

From definition (48), $Z = \sum_{q\in Q} \exp\left(-fX_0[q]e_0 + ...\right)$. Therefore,

$$S = -\frac{1}{Z} \sum_{q\in Q} \left(-fX_0[q]e_0 + ...\right) \exp\left(-fX_0[q]e_0 + ...\right) + \ln Z \tag{66}$$

From definition (47), the average of a quantity $\langle X_i e_i \rangle = \sum_{q\in Q} X_i[q] e_i \rho(q)$. Therefore,

$$S = f\langle X_0 e_0 \rangle + \langle ... \rangle + \ln Z \tag{67}$$

□

Definition 6 (Geometric equation of state)**.** *The equation of state of the ensemble of events is:*

$$d\hat{S} = \underbrace{fd\langle X_0 e_0 \rangle + ... + fd\langle X_{n-1} e_{n-1} \rangle}_{\text{time terms}} + \underbrace{\tilde{k}d\langle X_n e_n \rangle + ... + \tilde{k}d\langle X_{n+m-1} e_{n+m-1} \rangle}_{\text{space terms}} \tag{68}$$

2.1 *Exploration: From geometric entropy to scalar entropy*

Definition 7 (Von Neumann entropy)**.** *One can define a transformation on $\hat{S}$ such that $\hat{S} \to S \in \mathbb{R}_{\geq 0}$. The suggestion by Von Neumann is to define the entropy not by the usual Shannon/Boltzmann definition (as we have done here) but instead in reference to the choice of basis associated with the minimal value for the entropy for S. To do so, one first notices that the entropy $\hat{S}$ depends on the choice of basis. Then, one rotates the basis until one finds the lowest value for the entropy. Finally, one redefines the entropy to be this minimal value instead of the previous definition. It is possible to show (proof omitted) that this is equivalent to diagonalizing the density matrix and then tracing over it according to the typical definition of the Von Neumann entropy.*

$$S = \text{Tr}\hat{\rho} \ln \hat{\rho} \tag{69}$$

The obvious weakness with this approach is that the resulting entropy is not the full and complete entropy of the initial system! The resulting Von Neumann entropy is instead an entropy defined according to, roughly, the departure away from a pure quantum state. Consequently, one who defines the entropy using the Shannon entropy (as we have done) will obtain a greater value for the entropy of the system than the value obtained by the Von Neumann entropy.

Thus, in practice, one ought to be able to extract more information from a quantum system than the Von Neumann entropy suggests (and indeed one can). So, what information can one extract from a quantum system that is missing from the Von Neumann entropy? This will become clear with this next definition.

Definition 8 (Measurement entropy)**.** *The Shannon entropy defines the amount of information one gains by knowing which element q has been randomly select from a set $\mathbb{Q}$ under some probability distribution $\rho(q)$. In base e, it is defined as:*

$$S = -\sum_{q\in\mathbb{Q}} \rho(q) \ln \rho(q) \tag{70}$$

Suppose a quantum system defined by a function:

$$|\psi\rangle = \sum_{u\in\mathbb{U}} a(u) |u\rangle \tag{71}$$

where $\mathbb{U}$ is an eigenbasis of $|\psi\rangle$, and where $a(u)$ returns the probability amplitude associated with the eigenstate $|u\rangle$.

Then, recall that a quantum measurement is defined as a projection P

$$P(u) |\psi\rangle \to |u\rangle \text{ , with probability } a(u)a(u)^* \tag{72}$$

where $a(u)^*$ is the complex conjugate of $a(u)$, such that after measurement the system $|\psi\rangle$ finds itself in one eigenstate $|u\rangle$ randomly selected from the set $\mathbb{U}$ according to a probability of $a(u)a(u)^*$.

Since the measurement involves the random selection of one element from a set, one can then use the Shannon entropy to associate an entropy to the measurement. In this case,

$$S = -\sum_{u\in\mathbb{U}} a(u)a(u)^* \ln [a(u)a(u)^*] \tag{73}$$

This entropy corresponds to the amount of information an experimentalist gains by measuring a quantum system $|\psi\rangle$ within some eigenbasis $\mathbb{U}$.

The measurement entropy represents the "information distance" between the system and the final measurement outcome within some chosen basis.

We note that the Von Neumann entropy, unlike the measurement entropy, does not associate a gain of information to the measurement process. Indeed, consider the measurement of a quantum system in a pure state before and after the measurement, say from $\psi_{\text{before}} = \alpha |0\rangle + \beta |1\rangle$ to $\psi_{\text{after}} = |0\rangle$. The Von Neumann entropy, quantifying the departure from a pure state, is thus zero both before and after measurement. Both ψ_{before} and ψ_{after} are pure states.

We note that the measurement entropy, unlike the Von Neumann entropy, has the property that it is dependant on the choice of eigenbasis, and therefore the entropy is not invariant with respect to a unitary transformation U on $|\psi\rangle$.

Theorem 3. *Let us show that the measurement entropy is not invariant with respect to the unitary transformation.*

Proof. Suppose a quantum system:

$$|\psi\rangle = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{74}$$

where $\alpha\alpha^* + \beta\beta^* = 1$. The measurement entropy along the basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is

$$S = \alpha\alpha^* \ln [\alpha\alpha^*] + \beta\beta^* \ln [\beta\beta^*] \tag{75}$$

After a rotation of basis by the action of a unitary operator U, defined as

$$U = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix}, \text{ where } a, b \in \mathbb{C} \text{ and } aa^* + bb^* = 1 \tag{76}$$

Then,

$$U|\psi\rangle = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{77}$$

$$= (\alpha a - \beta b^*) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + (\alpha b + \beta a^*) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{78}$$

and the entropy of a measurement within the rotated basis is now:

$$\begin{aligned} S = (\alpha a - \beta b^*)(\alpha a - \beta b^*)^* \ln [(\alpha a - \beta b^*)(\alpha a - \beta b^*)^*] + \\ (\alpha b + \beta a^*)(\alpha b + \beta a^*)^* \ln [(\alpha b + \beta a^*)(\alpha b + \beta a^*)^*] \end{aligned} \tag{79}$$

We note that S in equation (75) is not equal to S in equation (79). □

To define the measurement entropy one must chose a basis. But what if no basis is chosen (or at least not yet); can we define an entropy which is basis-invariant (like the Von Neumann entropy) whilst also aware of the possibly of an eventual basis-dependant information-yielding measurement (like the measurement entropy) — a.k.a the best of both worlds? The answer will be the interval entropy.

2.2 *Main result: Interval entropy*

Theorem 4 (Interval entropy). *Due to the peculiar non-commutative properties of the algebra of events, the multivector $\hat{S}$ becomes a real number simply by squaring the equation of state:*

$$(dS)^2 = \Bigg(\underbrace{fd\langle X_0 e_0\rangle + ... + fd\langle X_{n-1} e_{n-1}\rangle}_{\text{time terms}} + \underbrace{\tilde{k}d\langle X_n e_n\rangle + ... + \tilde{k}d\langle X_{n+m-1} e_{n+m-1}\rangle}_{\text{space terms}}\Bigg)^2 \tag{80}$$

Unlike redefining the entropy using the Von Neumann definition, squaring $d\hat{S}$ does not change the equation nor the entropy, yet it nonetheless erases base-specific information making $(dS)^2$ a scalar as desired.

To help us interpret the squared equation, we now divide each side of the above equation by $\tilde{k}^2$. Finally, we identify the term $\tilde{k}^{-2}(dS)^2$ as the space-time interval $(ds)^2$. Then, the equation of state is easily recognized as the interval of general relativity (the metric) in $m+n$ space.

$$\underbrace{(ds)^2}_{\text{interval}} = \Bigg(\underbrace{cd\langle X_0 e_0\rangle + ... + cd\langle X_{n-1} e_{n-1}\rangle}_{\text{time terms}} + \underbrace{d\langle X_n e_n\rangle + ... + d\langle X_{n+m-1} e_{n+m-1}\rangle}_{\text{space terms}}\Bigg)^2 \tag{81}$$

where $(ds)^2 = \tilde{k}^{-2}(dS)^2$.

Proof. Expanding the power of two of the right-hand side of the equation and rearranging, it is straightforward to recover the generalized metric;

$$\underbrace{\tilde{k}^{-2}(dS)^2}_{\text{entropic distance}} = \underbrace{(ds)^2}_{\text{interval}} = \sum_{\mu\nu} g_{\mu\nu} d\langle X_\mu\rangle d\langle X_\nu\rangle \tag{82}$$

□

3 *Discussion*

Conjecture 1 (Present). *The relation $(ds)^2 = \tilde{k}^{-2}(dS)^2$ equates the space-time interval to an entropy. We note that the entropy is constant on the hypersurface traced by light: $0 = (ds)^2 = \tilde{k}^{-2}(dS)^2$ (nullsurface). Since the entropy is constant on the nullsurface, then the entropic distance between all events on the surface is zero. This nullsurface is thus the present of the observer and it represents the set of all events informationally accessible to the observer.*

Conjecture 2 (Past/Future). *We note that*

$$(ds)^2 = \tilde{k}^{-2}(dS)^2 \implies ds = \pm\tilde{k}^{-1}dS \tag{83}$$

Splitting the $\pm$, we can then define the past as

$$ds = -\tilde{k}^{-1}dS \tag{84}$$

and the future as

$$ds = \tilde{k}^{-1} dS \tag{85}$$

Conjecture 3 (Growing block universe)**.** *The relation* $(ds)^2 = \tilde{k}^{-2}(dS)^2$ *quantifies the informational departure of an event from the present. The role of this entropy is to hide from an observer knowledge of events outside its present. This entropy is only eliminated if and when the gap between the observer and the entropically-distant event is reduced to* 0 *(that is, when the observers moves forward in time until the event occurs).*

Conjecture 4 (Random measurements events)**.** *The future departs informationally from the present due to a positive entropic distance* $(\tilde{k}^{-1}dS > 0)$*. Since the entropy represents the number of microscopic states compatible with the current macroscopic state of the system, an observer pondering about its future will conclude that multi possible futures appears possible, but would expect only one to be actual when the interval entropy is reduced to zero. Such an observer can quantifies its possible futures using the measurement entropy defined earlier.*

Conjecture 5 (Measurement agreement)**.** *Since the entropy is constant along the nullsurface traced by light in space-time, and such entropy represents quantum entropy including measurement entropy, all observers along this surface necessarily agree on the result of all measurement outcomes. Disagreement between observers regarding measurement outcomes is impossible on the nullsurface of the present, as all observers know (or can know) what all others have measured (there is no informational distance between events on the nullsurface).*

Conjecture 6 (Entropic speed of light)**.** *The speed of light, usually taken as an axiom in special relativity, is here emergent as the ratio of the Lagrange multipliers of the partition function* $c := f/\tilde{k}$*. The speed of light here fulfills a role similar to the role fulfilled by the temperature in standard thermodynamics. Essentially, the speed of light is the "temperature" of space-time. The speed of light is a property emergent from the random selection of events from a larger set under the principle of maximum entropy. The speed of light is constant in an ensemble of space-time events at equilibrium.*

4 Outlook - Extended events in C-space

Geometric thermodynamics can be extended to C-space (a.k.a extended special relativity [8]) fairly easily. In this case, the formalism can account for the description of extended elements such as surfaces and volumes, and their evolution in space-time. The quantum properties discussed thus far will of course be inherited within this extension, and therefore the theory becomes a quantum theory of extended objects in space-time.

[8] C Castroa and M Pavšicb. The extended relativity theory in clifford spaces. *Quantization in Astrophysics, Brownian Motion, and Supersymmetry*, page 97, 2007

To obtain this, one simply needs to define events using the full geometric freedom afforded by Clifford algebra by including bivectors $(e_0e_1, e_0e_2, e_1e_2, ..)$, trivectors $(e_0e_1e_2)$, etc. into the multivector describing an event. The algebraic definition of an event then becomes:

$$
\begin{aligned}
\vec{q} :=& a && \text{(scalar)} \\
&+ X_0 e_0 + X_1 e_1 + ... && \text{(vectors)} \\
&+ A_0 e_0 e_1 + A_1 e_0 e_2 + ... && \text{(areas)} \\
&+ V_0 e_0 e_1 e_2 + ... && \text{(volumes)} \\
&+ H_0 e_0 e_1 e_2 e_3 + ... && \text{(hyper-volumes)} \\
&+ etc... && (86)
\end{aligned}
$$

One can then use the extended event $\vec{q}$ to define the partition function of an ensemble of extended events, as:

$$Z = \sum_{q \in \mathbb{Q}} \exp(-c\vec{q}) \tag{87}$$

With equation of state:

$$d\hat{S} = c\langle \vec{q} d \rangle \tag{88}$$

By squaring the entropy, the non-commutative properties of the basis ensures that basis information is erased and the entropy is a scalar (this is true even with the higher dimensional basis). In this case, one obtains the interval of extended special relativity in C-space which is equated to the interval entropy:

$$(ds)^2 = \tilde{k}^{-2}(dS)^2 = (c\langle \vec{q} \rangle)^2 \tag{89}$$

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