A Final Tentative of The Proof of The ABC Conjecture - Case $c = a + 1$

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Abstract: In this paper, we consider the abc conjecture in the case $c = a + 1$. Firstly, we give the proof of the first conjecture that $c < \text{rad}^2(ac)$ using the polynomial functions. It is the key of the proof of the abc conjecture. Secondly, the proof of the abc conjecture is given for $\varepsilon \geq 1$, then for $\varepsilon \in [0,1[$ for the two cases: $c \leq \text{rad}(ac)$ and $c > \text{rad}(ac)$. We choose the constant $K(\varepsilon)$ as $K(\varepsilon) = e^{\left(\frac{1}{\varepsilon^2}\right)}$. A numerical example is presented.
A Proof of The Conjecture: $C < \text{rad}^2(ac)$

A Final Tentative of The Proof of The $ABC$ Conjecture - Case $c = a + 1$

To the memory of my Father who taught me arithmetic

To the memory of my colleague and friend Dr.Eng. Chedly Fezzani (1943-2019) for his important work in the field of Geodesy and the promotion of the Geographic Sciences in Africa

1. Introduction and notations

Let $a$ a positive integer, $a = \prod_i a_i^{\alpha_i}$, $a_i$ prime integers and $\alpha_i \geq 1$ positive integers. We call radical of $a$ the integer $\prod_i a_i$ noted by $\text{rad}(a)$. Then $a$ is written as:

$$a = \prod_i a_i^{\alpha_i} = \text{rad}(a) \cdot \prod_i a_i^{\alpha_i - 1}$$  \hspace{1cm} (1.1)

We note:

$$\mu_a = \prod_i a_i^{-\alpha_i} \implies a = \mu_a \cdot \text{rad}(a)$$  \hspace{1cm} (1.2)

The $abc$ conjecture was proposed independently in 1985 by David Masser of the University of Basel and Joseph Œsterlé of Pierre et Marie Curie University (Paris 6) ([1]). It describes the distribution of the prime factors of two integers with those of its sum. The definition of the $abc$ conjecture is given below:

**Conjecture 1.3. ($abc$ Conjecture):** Let $a, b, c$ positive integers relatively prime with $c = a + b$, then for each $\varepsilon > 0$, there exists $K(\varepsilon)$ such that:

$$c < K(\varepsilon) \cdot \text{rad}(abc)^{1+\varepsilon}$$  \hspace{1cm} (1.4)

We know that numerically, $\frac{\log c}{\log(\text{rad}(abc))} \leq 1.616751$ ([2]). A conjecture was proposed that $c < \text{rad}^2(abc)$ ([3]). Here we will give the proof of it, in the case $c = a + 1$, using a polynomial function.

**Conjecture 1.5.** Let $a, b, c$ positive integers relatively prime with $c = a + b$, then:

$$c < \text{rad}^2(abc) \implies \frac{\log c}{\log(\text{rad}(abc))} < 2$$  \hspace{1cm} (1.6)

This result, I think is the key to obtain a proof of the veracity of the $abc$ conjecture.

2. A Proof of the conjecture (1.5) Case $c = a + 1$

Let $a, c$ positive integers, relatively prime, with $c = a + 1$. If $c < \text{rad}(ac)$ then we obtain:

$$c < \text{rad}(ac) < \text{rad}^2(ac)$$  \hspace{1cm} (2.1)

and the condition (1.6) is verified.

In the following, we suppose that $c \geq \text{rad}(ac)$. 

2.1 Notations

We note:

\[ a = \prod_i a_i^{a_i} \implies \text{rad}(a) = \prod_i a_i, \mu_a = \prod_i a_i^{a_i-1}, i = 1, N_a \]  \hspace{1cm} (2.2)

\[ c = \prod_j c_j^{\beta_j} \implies \text{rad}(c) = \prod_j c_j, \mu_c = \prod_j c_j^{\beta_j-1}, j = 1, N_c \]  \hspace{1cm} (2.3)

with \( a_i, c_j \) prime integers and \( N_a, N_c, \alpha, \beta \geq 1 \) positive integers. Let:

\[ R = \text{rad}(a) \cdot \text{rad}(c) = \text{rad}(ac) \]  \hspace{1cm} (2.4)

\[ \mathcal{R}(x) = \prod_i (x + a_i)^2 \cdot \prod_j (x + c_j) \implies \mathcal{R}(x) > 0, \forall x \geq 0 \]  \hspace{1cm} (2.5)

\[ F(x) = \mathcal{R}(x) - \mu_c \]  \hspace{1cm} (2.6)

From the last equations we obtain:

\[ F(0) = \mathcal{R}(0) - \mu_c = \text{rad}^2(a) \cdot \text{rad}(c) - \mu_c \]  \hspace{1cm} (2.7)

Then, our main task is to prove that \( F(0) > 0 \implies R^2 > c \).

2.1.1 The Proof of \( c < \text{rad}^2(ac) \)

From the definition of the polynomial \( F(x) \), its degree is \( 2N_a + N_c \). We have:

1. \( \lim_{x \to +\infty} F(x) = +\infty \),
2. \( \lim_{x \to +\infty} \frac{F(x)}{x} = +\infty \), \( F \) is convex for \( x \) large,
3. if \( x_1 \) is the great real root of \( F(x) = 0 \), and from the points 1., 2. we deduce that \( F''(x_1^+) > 0 \),
4. if \( x_1 < 0 \), then \( F(0) > 0 \).

Let us study \( F'(x) \) and \( F''(x) \). We obtain:

\[ F'(x) = \mathcal{R}'(x) \]

\[ \mathcal{R}'(x) = \left[ \prod_i N_a(x + a_i)^2 \right]' \cdot \prod_j N_c(x + c_j) + \prod_i N_a(x + a_i)^2 \cdot \left[ \prod_j N_c(x + c_j) \right]' \implies \]

\[ \left[ \prod_i N_a(x + a_i)^2 \right]' = 2 \prod_i N_a(x + a_i)^2 \cdot \left( \sum \frac{1}{x + a_i} \right) \]

\[ \left[ \prod_j N_c(x + c_j) \right]' = \prod_j N_c(x + c_j) \left( \sum_{j=1}^{N_c} \frac{1}{x + c_j} \right) \implies \]

\[ \mathcal{R}'(x) = \mathcal{R}(x) \cdot \left( \sum_{i=1}^{N_a} \frac{2}{x + a_i} + \sum_{j=1}^{N_c} \frac{1}{x + c_j} \right) > 0, \forall x \geq 0 \]  \hspace{1cm} (2.8)

\[ F'(x) = \mathcal{R}' = \mathcal{R}(x) \left( \sum_{i=1}^{N_a} \frac{2}{x + a_i} + \sum_{j=1}^{N_c} \frac{1}{x + c_j} \right) > 0, \forall x > 0 \implies \]

\[ F'(0) = \mathcal{R}(0) \left( \sum_{i=1}^{N_a} \frac{2}{a_i} + \sum_{j=1}^{N_c} \frac{1}{c_j} \right) = \text{rad}^2(a) \cdot \text{rad}(c) \left( \sum_{i=1}^{N_a} \frac{2}{a_i} + \sum_{j=1}^{N_c} \frac{1}{c_j} \right) > 0 \]  \hspace{1cm} (2.9)
For $F''(x)$, we obtain:

$$F''(x) = \mathcal{R}'' = \mathcal{R}'(x) \left( \sum_{i} \frac{2}{x + a_i} + \sum_{j} \frac{1}{x + c_j} \right) - \mathcal{R}(x) \left( \sum_{i} \frac{2}{(x + a_i)^2} + \sum_{j} \frac{1}{(x + c_j)^2} \right) = (2.10)$$

$$F''(x) = \mathcal{R}(x) \left[ \left( \sum_{i} \frac{2}{x + a_i} + \sum_{j} \frac{1}{x + c_j} \right)^2 - \sum_{i} \frac{2}{(x + a_i)^2} - \sum_{j} \frac{1}{(x + c_j)^2} \right]$$

$$F''(x) > 0, \forall x \geq 0 \quad (2.11)$$

We obtain also that $F''(0) > 0$.

Before we attack the proof, we take an example as: $1 + 8 = 9 \implies c = 9, a = 8, b = 1$. We obtain $\text{rad}(a) = 2, \text{rad}(c) = 3, \alpha_c = 3, \mathcal{R} = \text{rad}(ac) = 2 \times 3 = 6 < (c = 9)$ and $c = 9$ verifies $c < (R^2 = 6^2 = 36)$. We write the polynomial $F(x) = (x + 1)^2(x + 3) - 3 = x^3 + 7x^2 + 16x + 9 > 0, \forall x > 0$. Then $F'(x) = 3x^2 + 14x + 16$, we verifies that $F'(x) = 0$ has not real roots and $F'(x) > 0, \forall x \in \mathbb{R}$. We have also $F''(x) = 6x + 14$. $F''(x) = 0 \implies x = -7/3 \approx -2.33 \implies F(-7/3) = -79/27 \approx -2.92$. The point $(-7/3, -79/27)$ is an inflexion point of the curve of $y = F(x)$. We deduce that the curve is convex for $x \geq -7/3$. Let us now find the roots of $F(x) = 0$. As the degree of $F$ is three, the number of the real roots are 1 or 3. As there is one inflexion point, we will find one real root.

### 2.2 The Resolution of $F(x) = 0$

We want to resolve:

$$F(x) = x^3 + 7x^2 + 16x + 9 = 0 \quad (2.12)$$

Let the change of variables $x = t - 7/3$, the equation (2.12) becomes:

$$t^3 - \frac{79}{27} = 0 \quad (2.13)$$

For the resolution of (2.13), we introduce two unknowns:

$$t = u + v \implies (u + v)(3uv - \frac{1}{3}) + u^3 + v^3 - \frac{79}{27} = 0 \implies$$

$$\begin{cases} u^3 + v^3 = \frac{79}{3^3} \\ uv = \frac{1}{3^2} \end{cases} \quad (2.14)$$

Then $u^3, v^3$ are solutions of the equation:

$$X^2 - \frac{79}{3^3}X + \frac{1}{3^6} = 0 \quad (2.15)$$

and given below:

$$u^3 = \frac{1}{2} \left( 79 + 9\sqrt{77} \right) \implies$$

$$\begin{cases} u_1 = \sqrt[3]{\frac{1}{2} \left( \frac{79 + 9\sqrt{77}}{3^3} \right)} \approx 0.97515 \\ u_2 = j.u_1, \quad j = -\frac{1 + \sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \\ u_3 = j^2u_1 = j.u_1 \end{cases}$$

$$4$$
\[ v^3 = \frac{1}{2} \cdot \frac{79 - 9\sqrt{77}}{3^3} \] 
\[ \Longrightarrow \begin{cases} 
  v_1 = \sqrt[3]{\frac{1}{2} \left( \frac{79 - 9\sqrt{77}}{3^3} \right)} \approx 0.00016 \\
  v_2 = f^2 \cdot v_1 = f \cdot v_1 \\
  v_3 = f \cdot v_1 
\end{cases} \]
\[ v_3 = f \cdot v_1 \]
\[ (2.16) \]

Finally, taking into account the second condition of (2.14), we obtain the real root of (2.13):
\[ t = u_1 + v_1 = \sqrt[3]{\frac{1}{2} (\frac{79 + 9\sqrt{77}}{3^3}) + \frac{1}{2} (\frac{79 - 9\sqrt{77}}{3^3})} \approx 0.97531 \]
\[ x_1 = \frac{t - 7}{3} \approx \frac{1.35802}{3} \approx -1.35802 \]
\[ (2.17) \]

Then the first root of \( F(x) = 0 \) is \( x_1 \approx -1.358 < 0 \), the correction to the first root of \( \mathcal{R}(x) = (x + 2)^2(x + 3) = 0 \) is \( dx = x_1 - (-2) = -1.358 - (-2) = +0.642 \). As in our example \( F'(x) > 0 \), the function \( F(x) \) is an increasing function having a parabolic branch as \( x \to +\infty \), the curve \( y = F(x) \) intersects the line \( x = 0 \) in the half-plane \( y \geq 0 \to F(0) > 0 \to c < \text{rad}^2(ac) \) which is verified numerically.

### 2.3 The General Case

Let us return to the general case \( c = a + 1 \). We denote \( q = \min(a_i, c_j) \). If we consider that \( F(x) = \mathcal{R}(x) \), the equation \( F(x) = 0 \to \mathcal{R}(x) = 0 \) and the first real root is \( x_1 = -q \), the product of all the roots is \( P = \prod_i (x_i) \cdot \prod_j (x_j) = (-1)^{2N_c + N} \prod_i (a_i) \cdot \prod_j (c_j) \). But \( F(x) = \mathcal{R}(x) - \mu_c \), the constant coefficient of \( F(x) \) will be \( \prod_i (a_i)^2 \cdot \prod_j (c_j) - \mu_c \). The new product of the roots is \( F' = \prod_i (x_i')^2 \cdot \prod_j (x_j') = (-1)^{2N_c + N} (\prod_i (a_i) \cdot \prod_j (c_j) - \mu_c) \). The first root \( x_1 = -q \) becomes \( x_1' = -q + dx \). To estimate \( dx \), we can write to the order two that:
\[ F(-q + dx) = \mathcal{R}(-q + dx) - \mu_c = 0 \to \mathcal{R}(-q + dx) = \mu_c \to \mathcal{R}(-q) + dx \cdot \mathcal{R}'(-q) + \frac{dx^2}{2} \mathcal{R}''(-q) = \mu_c \]
\[ (2.18) \]

Supposing that \( a_1 = q = \min(a_i, c_j) \), from the equations (2.5-2.8-2.10), we have:
\[ \mathcal{R}(-a_1) = 0 \]
\[ \mathcal{R}'(-a_1) = 0 \]
\[ \mathcal{R}''(-a_1) = 2 \prod_{i=2}^{N_c} (a_i - a_1)^2 \cdot \prod_{j=1}^{N} (c_j - a_1) > 0 \to dx^2 = \frac{\mu_c}{\prod_{i=2}^{N_c} (a_i - a_1)^2 \cdot \prod_{j=1}^{N} (c_j - a_1)} \]
\[ (2.19) \]

We suppose that \( c > \text{rad}^2(ac) \to \mu_c > \text{rad}^2(a) \cdot \text{rad}(c) \to \mu_c > \mathcal{R}(0) \). We deduce that \( F(0) < 0 \) and \( x_1' = -a_1 + dx > 0 \to dx > 0 \). We take the positive value of \( dx \), then we obtain:
\[ dx = \frac{\sqrt{\mu_c}}{\prod_{i=2}^{N_c} (a_i - a_1) \cdot \prod_{j=1}^{N} (c_j - a_1)} \]
\[ (2.20) \]
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But $\mu_c = R(x'_1) = \prod_{i=1}^{N_x}(x'_1 + a_i)^2 \prod_{j=1}^{N_y}(x'_1 + c_j)$, we can write:

$$\mu_c = dx^2 \prod_{i=2}^{N_x}(dx + a_i - a_1)^2 \prod_{j=1}^{N_y}(dx + c_j - a_1) \implies \mu_c > dx^2 \prod_{i=2}^{N_x}(a_i - a_1)^2 \prod_{j=1}^{N_y}(c_j - a_1) \quad (2.21)$$

because all the terms $a_i - a_1$ and $c_j - a_1$ are positive numbers. Using the last inequality and the expression of $dx$ given by the equation (2.20), we obtain:

$$\mu_c > \frac{\mu_c}{\prod_{i=2}^{N_x}(a_i - a_1)^2 \prod_{j=1}^{N_y}(c_j - a_1)} \implies \prod_{i=2}^{N_x}(a_i - a_1) = \prod_{j=1}^{N_y}(c_j - a_1)$$

$$1 > 1 \implies \text{the contradiction} \implies \mu_c < \text{rad}^2(a) \text{rad}(c) \quad (2.22)$$

So, our supposition that $c > \text{rad}^2(ac)$ is false and we obtain the important result that $c < \text{rad}^2(ac)$ and the conjecture (1.5) is verified.

2.3.1 Examples

In this section, we are going to verify the above remarks with a numerical example. The example is given by:

$$1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6$$

$$\text{rad}(a) = 2 \times 3 \times 5 \times 7 \times 127 = 26670$$

$$\text{rad}(c) = 19$$

$$c = 19^5 = 47045881, \quad \mu_c = 19^5 = 2476099 \quad (2.23)$$

Using the notations of the paper, we obtain:

$$R(x) = (x + 2)^2(x + 3)^2(x + 5)^2(x + 7)^2(x + 127)^2(x + 19)$$

$$F(x) = R(x) - \mu_c$$

Let $X = x + 2$, the expression of $R(x)$ becomes:

$$\bar{R}(X) = X^2(X + 1)^2(X + 3)^2(X + 5)^2(X + 125)^2(X + 17)$$

The calculations give:

$$\bar{R}(X) = X^{11} + 285.X^{10} + 24808.X^9 + 657728.X^8 + 7424722.X^7 + 42772898.X^6$$

$$+ 134002080.X^5 + 223508940.X^4 + 187753125.X^3 + 597656251.X^2 \quad (2.24)$$

We want to estimate the first root of $F(x) = 0$, we write:

$$\bar{R}(X) - \mu_c = 0 \implies$$

$$X^{11} + 285.X^{10} + 24808.X^9 + 657728.X^8 + 7424722.X^7 + 42772898.X^6$$

$$+ 134002080.X^5 + 223508940.X^4 + 187753125.X^3 + 597656251.X^2 - 2476099 = 0 \quad (2.25)$$
If \( x = -2 \Rightarrow X = 0 \Rightarrow \mathcal{K}(X) - \mu_c < 0 \). If we take \( x_1 = -1.936315 \Rightarrow X_1 = 0.03685 \), then we obtain that:

\[
\mathcal{K}(x_1) - \mu_c = \mathcal{K}(X_1) - \mu_c \approx 177.82 > 0 \quad (2.26)
\]

Then, \( \exists \xi \) with \(-2 < \xi < x_1\) so that \( X' = 2 + \xi \) verifies \( \mathcal{K}(X') - \mu_c = 0 \) and \( \xi \) is the first root of \( F(x) = 0 \) and \( \xi < 0 \Rightarrow F(0) > 0 \Rightarrow \text{rad}^2(a)\text{rad}(c) - \mu_c > 0 \Rightarrow R^2 > c \) that is true. We have also \( \xi = -2 + dx = a_1 + dx \) and \( 0 < dx < a_1 \).

3. **The Proof of The ABC Conjecture (1.3) Case:** \( c = a + 1 \)

We denote \( R = \text{rad}(ac) \).

3.1 **Case:** \( \epsilon \geq 1 \)

Using the result of the theorem above, we have \( \forall \epsilon \geq 1 \):

\[
c < R^2 \leq R^{1+\epsilon} < K(\epsilon) \cdot R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(1 / \epsilon^2\right)}, \quad \epsilon \geq 1 \quad (3.1)
\]

We verify easily that \( K(\epsilon) > 1 \) for \( \epsilon \geq 1 \) and it is a decreasing function from the base of the neperian logarithm to 1.

3.2 **Case:** \( \epsilon < 1 \)

3.2.1 **Case:** \( c \leq R \)

In this case, we can write:

\[
c \leq R < R^{1+\epsilon} < K(\epsilon) \cdot R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(1 / \epsilon^2\right)}, \quad 0 < \epsilon < 1 \quad (3.2)
\]

Here also \( K(\epsilon) > 1 \) for \( \epsilon < 1 \) and the abc conjecture is true.

3.2.2 **Case:** \( c > R \)

In this case, we confirm that:

\[
c < K(\epsilon) \cdot R^{1+\epsilon}, \quad K(\epsilon) = e^{\left(1 / \epsilon^2\right)}, \quad 0 < \epsilon < 1 \quad (3.3)
\]

If not, then \( \exists \epsilon_0 \in ]0, 1[ \), so that the triplets \((a, 1, c)\) checking \( c > R \) and:

\[
c \geq R^{1+\epsilon_0} \cdot K(\epsilon_0) \quad (3.4)
\]

are in finite number. We have:

\[
c \geq R^{1+\epsilon_0} \cdot K(\epsilon_0) \Rightarrow R^{1-\epsilon_0} \cdot c \geq R^{1-\epsilon_0} \cdot R^{1+\epsilon_0} \cdot K(\epsilon_0) \Rightarrow R^{1-\epsilon_0} \cdot c \geq R^2 \cdot K(\epsilon_0) > c \cdot K(\epsilon_0) \Rightarrow R^{1-\epsilon_0} > K(\epsilon_0) \quad (3.5)
\]
As \( c > R \), we obtain:

\[
c^{1-\varepsilon_0} > R^{1-\varepsilon_0} > K(\varepsilon_0) \implies c^{1-\varepsilon_0} > K(\varepsilon_0) \implies c > K(\varepsilon_0) \left( \frac{1}{1-\varepsilon_0} \right)
\]

(3.6)

We deduce that it exists an infinity of triples \((a, 1, c)\) verifying (3.4), hence the contradiction. Then the proof of the abc conjecture in the case \( c = a + 1 \) is finished. We obtain that \( \forall \varepsilon > 0, c = a + 1 \) with \( a, c \) relatively coprime, \( 2 \leq a < c \):

\[
c < K(\varepsilon) \cdot \operatorname{rad}(ac)^{1+\varepsilon} \quad \text{with} \quad K(\varepsilon) = e \left( \frac{1}{e^2} \right)
\]

(3.7)

Q.E.D

4. Examples

In this section, we are going to verify some cases of one numerical example. The example is given by:

\[
1 + 5 \times 127 \times (2 \times 3 \times 7)^3 = 19^6
\]

(4.1)

\( a = 5 \times 127 \times (2 \times 3 \times 7)^3 = 47045880 \Rightarrow \mu_a = 2 \times 3 \times 7 = 42 \) and \( \operatorname{rad}(a) = 2 \times 3 \times 5 \times 7 \times 127 \),

\( b = 1 \Rightarrow \mu_b = 1 \) and \( \operatorname{rad}(b) = 1 \),

\( c = 19^6 = 47045880 \Rightarrow \operatorname{rad}(c) = 19 \). Then \( \operatorname{rad}(abc) = \operatorname{rad}(ac) = 2 \times 3 \times 5 \times 7 \times 19 \times 127 = 506730 \).

We have \( c > \operatorname{rad}(ac) \) but \( \operatorname{rad}^2(ac) = 506730^2 = 256775292900 > c = 47045880 \).

4.0.1 Case \( \varepsilon = 0.01 \)

\[
c < K(\varepsilon) \cdot \operatorname{rad}(ac)^{1+\varepsilon} \implies 47045880 < e^{10000} \cdot 506730^{1.01}. \quad \text{The expression of } K(\varepsilon) \text{ becomes:}
\]

\[
K(\varepsilon) = e^{\frac{\mu_a}{\operatorname{rad}(a)}} = e^{10000} = 8,747777149120053120152473488653e + 4342
\]

(4.2)

We deduce that \( c \ll K(0.01) \cdot 506730^{1.01} \) and the equation (3.7) is verified.

4.0.2 Case \( \varepsilon = 0.1 \)

\[
K(0.1) = e^{\frac{\mu_a}{\operatorname{rad}(a)}} = e^{100} = 2,6879363309671754205917012128876e + 43 \implies c < K(0.1) \times 506730^{1.01}.
\]

And the equation (3.7) is verified.

4.0.3 Case \( \varepsilon = 1 \)

\[
K(1) = e \implies c = 47045880 < e \cdot \operatorname{rad}^2(ac) = 697987143184, 212. \quad \text{and the equation (3.7) is verified.}
\]
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4.0.4 Case $\varepsilon = 100$

$$K(100) = e^{0.0001} \implies c = 47045880^2 < e^{0.0001} \cdot 506730^{101} = 1,522350248607608781853142687284e + 576$$

and the equation (3.7) is verified.

5. Conclusion

This is an elementary proof of the abc conjecture in the case $c = a + 1$. We can announce the important theorem:

**Theorem 1.** Let $a, c$ positive integers relatively prime with $c = a + 1$, $a \geq 2$ then for each $\varepsilon > 0$, there exists $K(\varepsilon)$ such that :

$$c < K(\varepsilon) \cdot \text{rad}(ac)^{1+\varepsilon} \quad (5.1)$$

where $K(\varepsilon)$ is a constant depending of $\varepsilon$ equal to $e \left( \frac{1}{\varepsilon^2} \right)$.

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**References**

