Fibonacci Motives

M. D. Sheppeard

Abstract

Motives are well connected to graphical techniques in quantum field theory. In motivic quantum gravity we consider categorical axioms, starting with the ribbon category of the Fibonacci anyon. Quantum logic dictates that the cardinality of a classical set is replaced by the dimension of a space. In this note we look at the geometry underlying Fibonacci numbers and apply it to the algebra of multiple zeta values.

1 The Fibonacci numbers

One often considers a positive integer $N \in \mathbb{N}$ as a product $e_1^{r_1} e_2^{r_2} \cdots e_k^{r_k}$ of k prime power factors. It is less well known (Zeckendorff's theorem) that N is uniquely a sum of nonconsecutive Fibonacci numbers. With the exception of $F_6 = 8$ and $F_{12} = 144$, every F_n greater than 1 contains a prime factor that is not in any lower F_m (Carmichael's theorem). The basic recursion rule is

$$F_{n+2} = F_{n+1} + F_n, \tag{1}$$

recovering the Fibonacci numbers for $F_1 = F_2 = 1$. The Lucas numbers have the same recursion rule, starting with $L_1 = 1$ and $L_2 = 3$. Then $L_{n-1} = F_n + F_{n-2}$.

Lemma 1: For primes p equal to 1 or 4 mod 5, p divides F_{p-1}. For 2 or 3 mod 5, p divides F_{p+1}. Lastly, 5 divides F_5.

Proposition 1: For every $n \in \mathbb{N}$, the sequence of F_i mod n is a cycle of length $L(n)$ (see Table 1) such that $p+1$ divides $L(n)$ when p is a factor of $n = 2, 3$ mod 5. Similarly, $p - 1$ divides $L(n)$ when $n = 1, 4$ mod 5.

Proof: A cycle requires a string $1, 0, 1$ in the sequence F_i mod n. By the lemma, for any prime p there exists an $F_m \equiv 0$ mod p, and the same is true for any n. If n divides F_m then it divides all F_{km} for $k \geq 2$, since the greatest common divisor of F_{km} and $F_{(k+1)m}$ is F_m, containing n. It remains to find the first $F_{t-1} \equiv 1$ mod n next to a zero. Cassini's rule

$$F_{t-1}F_{t+1} - F_t^2 = (-1)^t \tag{2}$$

has a right hand side equal to ± 1. In general, write $F_{t-1} = ln$ and $F_{t-2} = jn + 1 + x$ for $0 \leq x \leq n - 2$. Then the left hand side of (2) mod n is a square $(x+1)^2$. We can consider either a small x or a small $y = n - x$. When the right hand side equals -1, x cannot equal 0. When the right hand side equals

Table 1: F_n cycle lengths

n	2	3	5	6	7	8	9	10	12	13
$L(n)$	3	8	20	24	16	12	24	60	24	28

1, $x = 0$ as required. Since the zero places occur at multiples of $t - 1$, there are odd values for t, giving $x = 0$.

Up to mod 12, all cycles fit into one of length 240, increasing to 1680 at mod 13. The Cassini rule comes from the determinant of the modular group matrix T^t.

Our interest in the Fibonacci numbers comes from the importance of the ring of integers in $\mathbb{Q}(\sqrt{5})$ in quantum gravity. Here the half integers in dimension 8 are embedded densely [1] in $\mathbb{C}$ under the map $\mathbb{Z}^4/2 \to \mathbb{R}$

$$(a, b, c, d) \mapsto a + b\phi + c\rho + d\phi\rho, \tag{3}$$

where $\phi = (1 + \sqrt{5})/2$ is the golden ratio and $\rho = \sqrt{\phi + 2} = 5^{1/4}\sqrt{\phi}$. In the necessarily categorical axioms for gravity we cannot begin with $\mathbb{R}$ or $\mathbb{C}$, which are awkward in higher dimensional analogues of a topos [2][3], and do not correspond to motivic periods. In particular, ϕ is the limit of ratios F_{m+1}/F_m. What is the geometric information behind the Fibonacci numbers?

2 Golden word spaces

The ribbon category of the Fibonacci anyon [4][5] is universal [6] for quantum computation [7]. It's $SU(2)$ representation for the braid group B_3[8] is defined in terms of quaternion units J and K. Let

$$g = \exp^{7\pi J/10}, \quad f = J\phi + K\sqrt{\phi}, \quad h = fgf^{-1}. \tag{4}$$

Then $ghg = hgh$ is the braid relation. This is a rotation (by a 9° ν mixing angle) of the quaternion braid generators

$$\frac{1}{\sqrt{2}}(1 + J), \quad \frac{1}{\sqrt{2}}(1 + K). \tag{5}$$

A cyclic set I, J, K of three generators is associated to the $\mathbb{C} \otimes \mathbb{O}$ ideal algebra [9][10][11] for Standard model leptons and quarks in the Bilson-Thompson [12][13] ribbon scheme. Each ribbon strand on a particle diagram carries either a zero or 1/3 electric charge. Collecting the diagrams for a lepton pair and six quarks of positive charge, we obtain a parity cube

$$000, 00+, 0+0, +00, 0++, +0+, ++0, +++ \tag{6}$$

of charges on the vertices. The underlying permutation $(231) \in S_3$ is associated to the neutrino.

In this section we will obtain the parity cube in each dimension using the Fibonacci anyons. This qubit state space is extended to general qudits by subdividing the edges of a cube. For example, a square with halved edges carries three labels along each edge, defining all pure two qutrit states.

The integers $\mathbb{Z}$ are infinite dimensional, as follows. Each discrete dimension is labeled by a prime power path sequence: 1, e, e^2 etc. in the positive direction, so that the points of an infinite dimensional cubic lattice represent exactly $\mathbb{Z}$, following Pythagoras. Little cubes near the origin are the squarefree numbers. An auxilliary infinite string of 1s allows for affine words (inhomogeneous monomials) on the diagonal simplices. Now a point in $\text{Spec}(\mathbb{Z})$ is the infinite discrete space split into two by the orthogonal axis hyperplane. The whole infinite space is a vector space union (span) of all the hyperplanes. Dual to the set union of all hyperplanes is the infinite parity cube on quadrants.

Along with cube and permutohedra tiles we work with the associahedra. The fusion map for the anyon category is an associator arrow on the pentagon. Let $F(abcd)^y_x$ be a fusion coefficient for an internal edge y on the input tree and internal edge x in the set of allowed trees, with d labelling the root of a three leaved tree. Our anyon objects are 1 and τ, such that $\tau \circ \tau = 1 + \tau$. Following [4], the interesting coefficients satisfy the pentagon relation

$$\begin{aligned} F(\tau\tau c\tau)^d_a F(a\tau\tau\tau)^c_b = {} & F(\tau\tau\tau d)^c_1 F(\tau 1 \tau\tau)^d_b F(\tau\tau\tau b)^1_a \\ & + F(\tau\tau\tau\tau)^c_\tau F(\tau\tau\tau\tau)^d_b F(\tau\tau\tau b)^\tau_a. \end{aligned} \tag{7}$$

When $(abcd)$ contains a 1, the coefficients are 0 or 1. At $(abcd) = (\tau 11 \tau)$, we obtain $F(\tau\tau\tau\tau)^1_1 = (F(\tau\tau\tau\tau)^1_\tau)^2$. From $(abcd) = (1\tau 1\tau)$ it follows that $F(\tau\tau\tau\tau)^1_1 = -F(\tau\tau\tau\tau)^1_1$. Let $F(\tau\tau\tau\tau)^1_1 = -A$ and $F(\tau\tau\tau\tau)^1_\tau = i\sqrt{A}$. Then $(abcd) = (\tau\tau\tau\tau)$ gives $A^2 - A - 1 = 0$ with solution $A = -1/\phi$. In summary, the all-τ coefficients are

$$\begin{pmatrix} F^1_1 & F^\tau_1 \\ F^1_\tau & F^\tau_\tau \end{pmatrix} = \begin{pmatrix} \frac{1}{\phi} & \frac{i}{\sqrt{\phi}} \\ \frac{i}{\sqrt{\phi}} & \frac{-1}{\phi} \end{pmatrix}. \tag{8}$$

These appear in the B_3 representation

$$\sigma_1 = \begin{pmatrix} e^{-4\pi i/5} & 0 \\ 0 & e^{3\pi i/5} \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} \frac{e^{4\pi i/5}}{\phi} & \frac{e^{-3\pi i/5}}{\sqrt{\phi}} \\ \frac{e^{-3\pi i/5}}{\sqrt{\phi}} & \frac{-1}{\phi} \end{pmatrix}, \tag{9}$$

with phases from the hexagon rule. The phase in (4) comes from the difference of these phases.

Consider the number of fusion diagrams on d leaves when all inputs are set to τ and the bracketing is nested to the left. We write words in 1 and τ by following the internal edges from a leaf down to the root. Since all words start with τ, we omit this letter, leaving words of length $d-1$. For three leaves the words are $\tau 1$, 1τ and $\tau\tau$, counted by the Fibonacci number F_{d+1}, as given in Table 2. Figure 1 shows how these words are allocated to vertices of a parity cube in dimension t equal to the number of τ letters. The + parity marks the placement of a 1.

Table 2: internal edges for fusion

n	words
2	1, τ
3	1τ, $\tau 1$, $\tau\tau$
4	$\tau 1\tau$, $\tau\tau 1$, $1\tau\tau$, $\tau\tau\tau$, $1\tau 1$
5	$1\tau 1\tau$, $1\tau\tau 1$, $1\tau\tau\tau$, $\tau 1\tau 1$, $\tau\tau\tau\tau$, $\tau\tau\tau 1$, $\tau\tau 1\tau$, $\tau 1\tau\tau$

The number F_{d+1} is graded across cubes of different dimension,

$$F_{d+1} = \sum_{n=0}^{f(d/2)} \binom{d-n}{n}, \tag{10}$$

where $f(i)$ is the integer part. Consider the subset labels for cubic vertices [14], which appear in the octonion basis. The $+++$ target in Figure 1 is the word $e_1^3 e_2^3 e_3^3$, where the superscript denotes the dimension. The vertex $+00$ is $e_1^3 = 1\tau\tau$ and so on. Denote a source by e_0. Then F_5 counts the set

$$e_0^4, e_1^3, e_2^3, e_3^3, e_1^2 e_2^2. \tag{11}$$

Such words are often interpreted as differential forms, but we might think of them as numbers with prime factors e_i. Since the number of tree diagrams on d leaves is the Catalan number

$$C_d = \frac{1}{d}\binom{2(d-1)}{d-1}, \tag{12}$$

the total number of fusion trees is

$$N(d) = F_{d+1} C_d \quad \in 1, 2, 6, 25, 112, 546, \cdots \tag{13}$$

The number of strands d in a generic braid diagram for anyons is closely related to the M theory dimension in a higher algebra approach [15]. For example, in ten dimensions there is a 3-cube of electric charges and a 7-cube for magnetic information.

Let us now recall the connection [16][17] between knots and multiple zeta values [18]. From our perspective, the appearance of the golden ratio in a fusion map is reminiscent of the Drinfeld associator, with its infinite series of multiple zeta values. In the iterated integral form, a zeta argument is a word in two letters such that one letter only occurs as a singlet, much like the 1 in our fusion words.

3 Knots and motivic numbers

A multiple zeta value (MZV) is the unsigned case of the signed Euler sum

$$\zeta(n_1, n_2, n_3, \cdots, n_l; \sigma_1, \cdots, \sigma_l) = \sum_{k_i > k_{i+1} > 0} \frac{\sigma_1^{k_1} \sigma_2^{k_2} \cdots \sigma_l^{k_l}}{k_1^{n_1} k_2^{n_2} \cdots k_l^{n_l}} \tag{14}$$

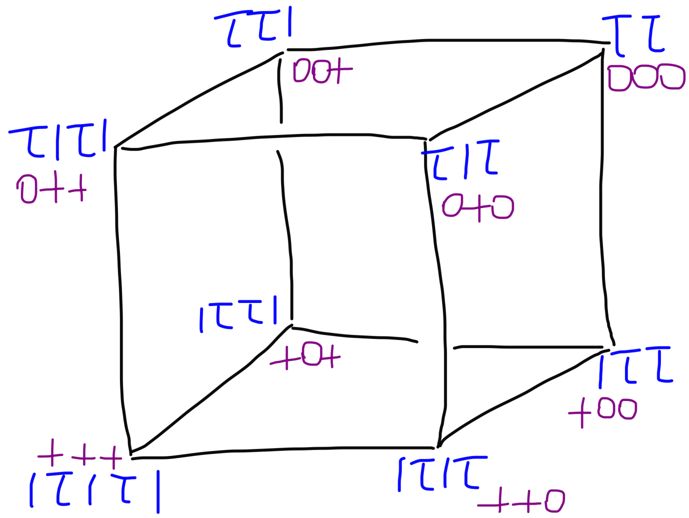

Figure 1: Fusion words

of *depth* l and *weight* $n = \sum_i n_i$, with $\sigma_i \in \pm 1$. Recall that the Mobius function $\mu(n)$ on $\mathbb{N}$ is zero on non square free n and $(-1)^r$ for r prime factors. The square free $n \in \mathbb{N}$ are the targets of undivided cubes in the infinite dimensional $\mathbb{Z}$. An MZV is irreducible if not expressed as a $\mathbb{Q}$ linear combination of other MZVs of the same weight. The number E_n of irreducible signed Euler sums of weight n is [16][17]

$$E_n = \frac{1}{n} \sum_{D|n} \mu(n/D) L_D = \frac{1}{n} \sum_{D|n} \mu(n/D)(F_{D-1} + F_{D-3}), \tag{15}$$

where L_D is the Lucas number. The number M_n of irreducible MZVs of weight n is the number of knots with n positive crossings (and no negative crossings). It's value replaces L_D by P_D, the Perrin number, satisfying the recursion

$$P_D = P_{D-2} + P_{D-3} \tag{16}$$

for $P_1 = 0$, $P_1 = 2$ and $P_3 = 3$.

An argument $(n_1, \cdots, n_l)$ of an MZV, such that only n_l may equal 1, is expressed as a word in two letters A and B, such that all words start with A and end with B, and B only occurs as a singleton. First reduce the argument to the ordinals $(n_1 - 1, n_2 - 1, \cdots, n_l - 1)$. The corresponding word is $A^{n_1-1} B A^{n_2-1} B \cdots A^{n_l-1} B$. Each copy of A is assigned the form dz/z and each B the form $dz/(1-z)$ in the iterated integral expression for the MZV. For example

$$\zeta(3,1) = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{dz}{z} \frac{dz}{z} \frac{dz}{1-z} \frac{dz}{1-z}. \tag{17}$$

Compare this to our golden words, with the additional τ at the start of every word, and a 1 occuring only as a singleton. Add a 1 at the end of every allowed word, to obtain precisely a set of MZV words. This extra 1 adds a bigon piece to the root edge of the polygon that is being chorded by a dual tree. The length of the internal word is essentially the weight,

$$\sum n_i - 2 = n - 2 = d - 1, \tag{18}$$

where d is the number of leaves on the fusion tree. Thus the weight $d+1$ is associated to braids in the category on d strands, but fewer than d strands may be used to draw a knot.

An example of a positive knot with n crossings and $n-1$ strands is the trefoil knot σ_1^3 in B_2. It corresponds to $\zeta(3)$, from the internal word τ on two leaves. The word 1 on two leaves gives $\zeta(2,1)$. Other torus knots of type $(2k+1,2)$ define the zeta values $\zeta(2k+1)$ [16]. At three leaves, the MZVs are $\zeta(3,1)$, $\zeta(4)$ and $\zeta(2,2)$. The total number of MZVs of weight $d+1$ is F_{d+1}, and the recursion $F_d + F_{d-1}$ splits F_{d+1} into words ending in either τ or 1 respectively.

Now recall that renormalisation in quantum field theory relies on the symmetric Hopf algebra of labeled rooted trees [20][21]. Our internal fusion words label a generic corolla tree with d leaves, which is a building block for symmetric trees. By restricting to the F_d words that end in τ, we ensure that the grafting of little corollas onto other trees is always possible. This suggests the F_{D-1} term in (15). F_{D-3} counts the number of internal words ending in $\tau 1$ and starting with τ. Thus L_D only excludes words that begin and end with 1, the so called vacuum words.

The square free divisors D of $d+1$ pick out a parity cube at the origin in dimension k inside $\mathbb{Z}$, where k is the number of prime factors in $d+1$. This cube has 2^k vertices, each marked with a Lucas number L_D in (15), for D a word in the e_i and a target word $e_1 e_2 \cdots e_k$. The signs of μ alternate on parity for the differential forms in e_i.

Values of M_n correspond to full words with even clusters of τ letters, corresponding to odd arguments for MZVs, as proved in [19]. This theorem requires an auxiliary five crossings to obtain the correct weight. For example, $M_{13} = 3$ [16] comes from fusion words on 7 strands, where the irreducibles are $\zeta(3,5)$, $\zeta(5,3)$ and $\zeta(7,1)$.

Consider the shuffle algebra for MZVs. The shuffle unit is the empty letter. The recursion law on A and B words is

$$\begin{aligned} l_1 l_2 \cdots l_u \cup k_1 k_2 \cdots k_v = {} & l_1(l_2 \cdots l_u \cup k_1 \cdots k_v) \\ & + k_1(l_1 \cdots l_u \cup k_2 \cdots k_v). \end{aligned} \tag{19}$$

The minimum zeta shuffle is

$$\zeta(2) \cup \zeta(2) = AB \cup AB = 2ABAB + 4AABB = 2\zeta(2,2) + 4\zeta(3,1). \tag{20}$$

For our fusion letter 1 we have $1 \cup 1 = 2 \cdot 1$ and $1^{\cup n} = 2^n \cdot 1$. The fusion vertex $\tau \circ \tau$ corresponds to $AAB + ABB = 2\zeta(3)$. Since $2\zeta(3)$ is also $\tau \cup \tau$, we have

$$\tau \circ \tau = \tau \cup \tau. \tag{21}$$

Elsewhere MZVs have been computed using the associahedra.

4 Comment

As category theorists, we are looking here at a functor from the category of finite sets (parity cubes) to a Fibonacci category. The intersection of two ordinals gives their greatest common divisor, which translates to the functorality of gcd on F_n. Primes e_i are in **Sets** (albeit not in the usual way) while the additive decomposition is in **F**.

In quantum gravity, universal cohomology does not begin with a consideration of classical geometries, which emerge from quantum computation. Compactified Minkowski space $SU(2) \times U(1)$ emerges from B_3 ribbons [6]. Observe the interplay here of different primes: the prime power axis labels (paths) or the truncation of a cubic lattice using qudits (vertex rule). The recovery of l-adic and other cohomologies is now studied by mathematicians using deformation parameters. Golden ring deformations are a basis for generic real and complex numbers. From this perspective, focusing on higher dimensional categorical axioms, we don't worry too much about the preponderence of infinite dimensional diagrams. Limits are easy to define and it all comes down to good choices for finite diagrams.

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