The Covariant Derivative Operator and Scalars

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Abstract

The writing delineates some peculiar aspects of the covariant operator. It appears that the metric coefficients have to disappear.

Introduction

The successive operations of two covariant derivative [1]operators on a tensor is usually non commutative. But there are many intricate issues involved in the issue. It seems that the metric coefficients have to vanish.

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Calculations

We consider the covariant derivative operator.For scalars in a torsion free field

$$\nabla_i \nabla_j f = \nabla_j \nabla_i f \quad (1)$$

Since $g^{\alpha\beta} P_\alpha Q_\beta$is a scalar we have

$$\nabla_i \nabla_j \left(g^{\alpha\beta} P_\alpha Q_\beta\right) = \nabla_j \nabla_i \left(g^{\alpha\beta} P_\alpha Q_\beta\right) \quad (2)$$

Since $\nabla_i g^{\alpha\beta} = 0$

$$g^{\alpha\beta} \nabla_i \nabla_j \left(P_\alpha Q_\beta\right) = g^{\alpha\beta} \nabla_j \nabla_i \left(P_\alpha Q_\beta\right)$$

$$\Rightarrow g^{\alpha\beta} \nabla_i \nabla_j \left(P_\alpha Q_\beta\right) - g^{\alpha\beta} \nabla_j \nabla_i \left(P_\alpha Q_\beta\right) = 0$$

$$\Rightarrow g^{\alpha\beta} \left[\nabla_i \nabla_j - \nabla_j \nabla_i\right]\left(P_\alpha Q_\beta\right) = 0 \; for\ arbitrary P_\alpha\ and\ Q_\beta \; (3)$$

We have,

$$\nabla_{\mathrm{i}} \nabla_j \left(P_\alpha Q_\beta\right) = \nabla_{\mathrm{i}} \left(P_\alpha \nabla_j Q_\beta + Q_\beta \nabla_j P_\alpha\right)$$

$$\nabla_{\mathrm{i}} \nabla_j \left(P_\alpha Q_\beta\right) = P_\alpha \nabla_{\mathrm{i}} \nabla_j Q_\beta + (\nabla_{\mathrm{i}} P_\alpha)\left(\nabla_j Q_\beta\right) + \left(\nabla_i Q_\beta\right)\left(\nabla_{\mathrm{j}} P_\alpha\right) + Q_\beta \nabla_{\mathrm{i}} \nabla_j P_\alpha \; (4)$$

$$\nabla_{\mathrm{j}}\nabla_i\left(P_\alpha Q_\beta\right) = \nabla_{\mathrm{j}}\left(P_\alpha \nabla_i Q_\beta + Q_\beta \nabla_i P_\alpha\right)$$

$$\nabla_{\mathrm{j}}\nabla_i\left(P_\alpha Q_\beta\right) = P_\alpha \nabla_{\mathrm{j}}\nabla_i Q_\beta + \left(\nabla_{\mathrm{j}} P_\alpha\right)\left(\nabla_i Q_\beta\right) + \left(\nabla_j Q_\beta\right)\left(\nabla_{\mathrm{i}} P_\alpha\right) + Q_\beta \nabla_{\mathrm{j}}\nabla_i P_\alpha \quad (5)$$

From (4) and (5)

$$\left[\nabla_{\mathrm{i}}\nabla_j - \nabla_{\mathrm{j}}\nabla_i\right]\left(P_\alpha Q_\beta\right) = P_\alpha\left(\nabla_{\mathrm{i}}\nabla_j - \nabla_{\mathrm{j}}\nabla_i\right)Q_\beta + Q_\beta\left(\nabla_{\mathrm{i}}\nabla_j - \nabla_{\mathrm{j}}\nabla_i\right)P_\alpha (6)$$

Using (3)

$$g^{\alpha\beta}\left[\nabla_{\mathrm{i}}\nabla_j - \nabla_{\mathrm{j}}\nabla_i\right]\left(P_\alpha Q_\beta\right) = g^{\alpha\beta} P_\alpha\left(\nabla_{\mathrm{i}}\nabla_j - \nabla_{\mathrm{j}}\nabla_i\right)Q_\beta + g^{\alpha\beta} Q_\beta\left(\nabla_{\mathrm{i}}\nabla_j - \nabla_{\mathrm{j}}\nabla_i\right)P_\alpha = 0 \quad (7)$$

Next we apply the following formula[2] on (7)

$$\left[\nabla_{\mathrm{i}}\nabla_j - \nabla_{\mathrm{j}}\nabla_i\right]A_p = R^n{}_{pij} A_n (8)$$

$$g^{\alpha\beta} P_\alpha R^n{}_{\beta ij} Q_n + g^{\alpha\beta} Q_\beta R^n{}_{\alpha ij} P_n = 0 \; for\ arbitray P_\alpha, Q_\beta \; (9)$$

$$g^{\alpha\beta} P_\alpha R^0{}_{\beta ij} Q_0 + \left[g^{\alpha\beta} P_\alpha R^k{}_{\beta ij} Q_k\right]_{k=1,2,3} + g^{\alpha 0} Q_0 R^n{}_{\alpha ij} P_n + \left[g^{\alpha k} Q_k R^n{}_{\alpha ij} P_n\right]_{k=1,2,3} = 0 \quad (10)$$

Since P_α and Q_β are arbitrary we make Q_0 five times its previous value

$$5g^{\alpha\beta} P_\alpha R^0{}_{\beta ij} Q_0 + \left[g^{\alpha\beta} P_\alpha R^k{}_{\beta ij} Q_k\right]_{k=1,2,3} + 5g^{\alpha 0} Q_0 R^n{}_{\alpha ij} P_n + \left[g^{\alpha k} Q_k R^n{}_{\alpha ij} P_n\right]_{k=1,2,3} = 0 \quad (11)$$

Subtracting (10) from (11)

$$g^{\alpha\beta} P_\alpha R^0{}_{\beta ij} Q_0 + g^{\alpha 0} Q_0 R^n{}_{\alpha ij} P_n = 0$$

$$g^{\alpha\beta} P_\alpha R^0{}_{\beta ij} + g^{\alpha 0} R^n{}_{\alpha ij} P_n = 0 \quad (12)$$

We expand (12) to write

$$g^{0\beta} P_0 R^0{}_{\beta ij} + g^{k\beta} P_\alpha R^0{}_{kij} + g^{\alpha 0} R^0{}_{\alpha ij} P_0 + g^{\alpha 0} R^k{}_{\alpha ij} P_k = 0 \; (13)$$

Setting

$$P_0 \to 500 P_0$$

$$500 g^{0\beta} P_0 R^0{}_{\beta ij} + g^{k\beta} P_\alpha R^0{}_{kij} + 500 g^{\alpha 0} R^0{}_{\alpha ij} P_0 + g^{\alpha 0} R^k{}_{\alpha ij} P_k = 0 (14)$$

]Takong the difference between (14) and (13) we obtain

$$499 g^{0\beta} P_0 R^0{}_{\beta ij} + 499 g^{\alpha 0} R^0{}_{\alpha ij} P_0 = 0$$

$$g^{0\beta} R^0{}_{\beta ij} + g^{\alpha 0} R^0{}_{\alpha ij} = 0 \; (15)$$

We could make similar type of adjustments with other components sometimes changing all of them simultaneously in different proportions.

The only solution would be to have$g^{\alpha\beta} = 0$

A closer Look at a Formula

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = A_{\alpha\beta}\nabla_\gamma B^{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta} \ \ (16)$$

Proof:

We consider the following relations

$$\nabla_\gamma A^{\alpha\beta} = A^{\alpha\beta};_\gamma = \frac{\partial A^{\alpha\beta}}{\partial x^\gamma} + \Gamma_{\gamma s}{}^{\alpha} A^{s\beta} + \Gamma_{\gamma s}{}^{\beta} A^{\alpha s}$$

$$\nabla_\gamma B_{\alpha\beta} = B_{\alpha\beta};_\gamma = \frac{\partial B_{\alpha\beta}}{\partial x^\gamma} + \Gamma^{s}{}_{\gamma\alpha} B_{s\beta} + \Gamma^{s}{}_{\gamma\beta} B_{\alpha s}$$

[The above relations do not assume $A^{\alpha\beta}$ and $B_{\alpha\beta}$ as symmetric tensors]

We obtain,

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = B_{\alpha\beta}\left(\nabla_\gamma A^{\alpha\beta} - \Gamma_{\gamma s}{}^{\alpha}A^{s\beta} - \Gamma_{\gamma s}{}^{\beta}A^{\alpha s}\right) + A^{\alpha\beta}\left(\nabla_\gamma B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha}B_{s\beta} + \Gamma^{s}{}_{\gamma\beta}B_{\alpha s}\right)$$

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = B_{\alpha\beta}\left(-\Gamma_{\gamma s}{}^{\alpha}A^{s\beta} - \Gamma_{\gamma s}{}^{\beta}A^{\alpha s}\right) + A^{\alpha\beta}\left(\Gamma^{s}{}_{\gamma\alpha}B_{s\beta} + \Gamma^{s}{}_{\gamma\beta}B_{\alpha s}\right) + A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta}$$

$$= -\Gamma_{\gamma s}{}^{\alpha}g^{s\beta}B_{\alpha\beta} - \Gamma_{\gamma s}{}^{\beta}g^{\alpha s}B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha}A^{\alpha\beta}T_{s\beta} + \Gamma^{s}{}_{\gamma\beta}A^{\alpha\beta}B_{s\alpha} + A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta}$$

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = \left(-\Gamma_{\gamma s}{}^{\alpha}A^{s\beta}B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha}A^{\alpha\beta}B_{s\beta}\right) + \left(\Gamma^{s}{}_{\gamma\beta}A^{\alpha\beta}B_{\alpha s} - \Gamma_{\gamma s}{}^{\beta}A^{\alpha s}B_{\alpha\beta}\right) + A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta} \ \ (2)$$

[In the above α, s, β are dummy indices]

We work out the two parentheses separately.

With the second term in the first parenthesis to the right we interchange as follows

$$\alpha \leftrightarrow s$$

$$\left(-\Gamma_{\gamma s}{}^{\alpha}A^{s\beta}B_{\alpha\beta} + \Gamma^{s}{}_{\gamma\alpha}A^{\alpha\beta}B_{s\beta}\right) = \left(-\Gamma_{\gamma s}{}^{\alpha}A^{s\beta}T_{\alpha\beta} + \Gamma^{\alpha}{}_{\gamma s}A^{s\beta}B_{\alpha\beta}\right) = 0$$

We do not have to worry about reflections on the left side of (5)because alpha and beta on the left side also disappear on contraction.

Indeed recalling (2) and using the relation:$B_{\alpha\beta}A^{\alpha\beta} = B_{\mu\nu}A^{\mu\nu}$ we may rewrite it [equation (2)] in the following form :

$$\frac{\partial}{\partial x^\gamma}\left(B_{\mu\nu}A^{\mu\nu}\right) = B_{\alpha\beta}\left(\nabla_\gamma A^{\alpha\beta} - \Gamma_{\gamma s}{}^{\alpha} A^{s\beta} - \Gamma_{\gamma s}{}^{\beta} A^{\alpha s}\right) + A^{\alpha\beta}\left(\nabla_\gamma B_{\alpha\beta} + \Gamma^s{}_{\gamma\alpha} B_{s\beta} + \Gamma^s{}_{\gamma\beta} B_{\alpha s}\right) + A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta}$$

There is no α, β on the left side of the above.

With the second term in the second parenthesis

$$\beta \leftrightarrow s$$

$$\left(\Gamma^s{}_{\gamma\beta} A^{\alpha\beta} B_{\alpha s} - \Gamma_{\gamma s}{}^{\beta} A^{\alpha s} B_{\alpha\beta}\right) = \left(\Gamma^s{}_{\gamma\beta} A^{\alpha\beta} B_{\alpha s} - \Gamma_{\gamma\beta}{}^{s} A^{\alpha\beta} B_{\alpha s}\right) = 0$$

$$\frac{\partial}{\partial x^\gamma}\left(B_{\alpha\beta}A^{\alpha\beta}\right) = A^{\alpha\beta}\nabla_\gamma B_{\alpha\beta} + B^{\alpha\beta}\nabla_\gamma A_{\alpha\beta}$$

Formula (16) is often used implicitly in mainstream literature[3]

Considering (3.1.20) from[3]

$$t^a \nabla_a\left(g_{bc} u^b w^c\right) = 0$$

Analyzing the above relation

$$\nabla_a\left(g_{bc} u^b w^c\right) \equiv \frac{\partial}{\partial x^a}\left(g_{bc} u^b w^c\right)$$

We have using (16)

$$\nabla_a\left(g_{bc} u^b w^c\right) = g_{bc}\nabla_a\left(u^b w^c\right) + u^b w^c \nabla_a g_{bc}$$

etc. etc.

Alternative Considerations

We consider two scalars $\chi(t,x,y,z)$ and $\psi(x,y,z,t)$on the same manifold. For every point (t,x,y,z) we choose another point (t',x',y',z') *on the same manifold* such that $\chi(t,x,y,z) = \psi(x',y',z',t')$

Next we choose a transformation $(t,x,y,z) \to (x',y',z',t')$in order to create a transformed manifold

Automatically

$$\chi(t,x,y,z) = \chi(t((x',y',z',t')), x(x',y',z',t'), y(x',y',z',t'), z(x',y',z',t'))$$

Again by our chosen transformation

$$\chi(t,x,y,z) = \psi(x',y',z',t')$$

Therefore on the transformed manifold,

$$\chi(t((x',y',z',t')), x(x',y',z',t'), y(x',y',z',t'), z(x',y',z',t')) = \psi(x',y',z',t')$$

On our chosen manifold [transformed manifold]

$$\chi(t((x',y',z',t')), x(x',y',z',t'), y(x',y',z',t'), z(x',y',z',t')) - \psi(x',y',z',t') = 0$$

Finally we set $\chi(t,x,y,z) = \frac{8\pi G}{c^4} g_{\alpha\beta} T^{\alpha\beta}$ and $\psi(t,x,y,z) = g_{\alpha\beta} T^{\alpha\beta}$

[$g_{\alpha\beta}$:metric tensor; $T^{\alpha\beta}$ stress energy tensor]

We have

$$\frac{8\pi G}{c^4} g_{\alpha\beta} T^{\alpha\beta} = g_{\alpha\beta} T^{\alpha\beta}$$

$$-R = R \Rightarrow 2R = 0 \Rightarrow R = 0$$

Dot Product Preserving Transport

In parallel transport[4] the two vectors the transported parallel to themselves.In dot product preserving transport product the dot is preserved but the two vectors individually are not transported parallel to themselves.

We have due to the preservation of dot product,

$$t^i \nabla_i \left(g_{\alpha\beta} u^\alpha v^\beta \right) = 0 \ (17)$$

Since each vector is not transported parallel to itself we have

$$t^i \nabla_i u^\alpha \neq 0; t^i \nabla_i v^\beta \neq 0 \ \ (18)$$

.

We transform to a frame of reference where t^i has only one non zero component.

$t^{k'} \nabla_{k'} \left(g_{\alpha\beta}' u^{\alpha'} v^{\beta'} \right) = 0$[no summation on k': prime denotes the new frame of reference and not differentiation]

$$\nabla'_k \left(g_{\alpha\beta}' u^{\alpha'} v^{\beta'} \right) = 0 (19)$$

$$u^{\alpha'} v^{\beta'} \nabla_{i'} \left(g_{\alpha\beta}' \right) + g_{\alpha\beta}' \nabla_{i'} \left(u^{\alpha'} v^{\beta'} \right) = 0$$

Since $\nabla_i(g_{\alpha\beta}) = 0$,we have,

$$g_{\alpha\beta}{}'\nabla_i(u^{\alpha'}v^{\beta'}) = 0 \quad (20)$$

The vectors $u^{\alpha'}$and $v^{\beta}{}'$ and consequently their individual components are *arbitrary*. Therefore

$$g'_{\alpha\beta} = 0 \Rightarrow g_{\alpha\beta} = 0 \quad (21)$$

[the null tensor remains null in all frames of reference]That implies that the Riemann tensor , Ricci tensor and the Ricci scalar are all zero valued objects.

The Line Element and the Symmetric Nature of the Metric Coefficients

So long as we are on the same manifold, the line element is preserved. This is not true for distinct manifolds

Example: A room with a flat floor and a hemispherical roof is considered. A small arc is drawn on the roof and its projection is taken on the floor. With this transformation

$$ds'^2 \neq ds^2$$

Only if

$$ds'^2 = ds^2$$

then $g_{\mu\nu}$behaves as a rank two tensor. Indeed

$$ds'^2 = ds^2$$

$$\Rightarrow \bar{g}_{\mu\nu} d\bar{x}^\mu d\bar{x}^\nu = g_{\alpha\beta} dx^\alpha dx^\beta$$

$$= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} d\bar{x}^\mu \frac{\partial x^\beta}{\partial \bar{x}^\nu} d\bar{x}^\nu$$

$$= g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} d\bar{x}^\mu d\bar{x}^\nu$$

$$\Rightarrow \bar{g}_{\mu\nu} = g_{\alpha\beta} \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu}$$

We revisit the idea[7] that the metric tensor is a symmetric tensor. Indeed

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

By interchanging the dummy indices μ and ν we have,

$$ds^2 = g_{\nu\mu} dx^\nu dx^\mu$$

$$\Rightarrow ds^2 = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu})dx^\mu dx^\nu$$

$g_{\mu\nu} + g_{\nu\mu}$ is asymmetric quantity. The relation

$$g_{\mu\nu}dx^\mu dx^\nu = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu})dx^\mu dx^\nu$$

is true for arbitrary $dx^\mu$and dx^ν.Therefore

$$g_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} + g_{\nu\mu})$$

$$\Rightarrow g_{\mu\nu} = g_{\nu\mu}$$

For an *arbitrary* non singular transformation a symmetric tensor has to be produced. That is impossible unless the metric tensor is the null tensor. This corroborates our inference from the "Dot Product Preserving Transport.

Riemannian Curvature Tensor

If we analyze in terms of the general coordinate systems[orthogonal or non orthogonal]we shall use $\mathrm{R}_{\alpha\alpha\gamma\delta} = \mathrm{R}_{\alpha\beta\gamma\gamma} = 0$ for same components in all reference frames [and not $\mathrm{R}_{\alpha\beta\gamma\delta} = \mathrm{R}_{\alpha\gamma\beta\delta} = 0$]Possibly $\mathrm{R}_{\alpha\beta\alpha\delta}$, $\mathrm{R}_{\alpha\beta\gamma\alpha}$, $\mathrm{R}_{\alpha\beta\gamma\delta}$, $\mathrm{R}_{\alpha\gamma\beta\delta}$and $\mathrm{R}_{\alpha\beta\beta\alpha}$ are non zero

The zeros will occur [components] every time we transform to some other arbitrary reference frame. All the components have to mix in order to produce the zeros in the same positions. The transformation elements will also change as we select different frames of reference.

Zeros occurring in all reference is impossible unless $\mathrm{R}_{\alpha\beta\gamma}$ =0 for each α, β, γ and δ.

The Riemannian tensor being zero, the Ricci tensor is also a null tensor and the Ricci scalar stands zero.

That the Ricci tensor is zero has been proved by an alternative method towards the beginning of the section.

Conclusion

As mentioned earlier that there is some peculiarity about the covariant operators. Their behavior indicates that the metric coefficients have to disappear

References

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