The Covariant Operator

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Abstract

The writing delineates some peculiar aspects of the covariant operator. It appears that the metric coefficients have to disappear.

Introduction

The successive operations of two covariant derivative [1]operators on a tensor is usually non commutative. But there are many intricate issues involved in the issue. It seems that the metric coefficients have to vanish.

Calculations

For scalars in a torsion free field

$$\nabla_i \nabla_j f = \nabla_j \nabla_i f$$

Since $g^{\alpha\beta} P_\alpha Q_\beta$is a scalar we have

$$\nabla_i \nabla_j \left(g^{\alpha\beta} P_\alpha Q_\beta\right) = \nabla_j \nabla_i \left(g^{\alpha\beta} P_\alpha Q_\beta\right)$$

Since $\nabla_i g^{\alpha\beta} = 0$

$$g^{\alpha\beta} \nabla_i \nabla_j \left(P_\alpha Q_\beta\right) = g^{\alpha\beta} \nabla_j \nabla_i \left(P_\alpha Q_\beta\right)$$

$$\Rightarrow g^{\alpha\beta} \nabla_i \nabla_j \left(P_\alpha Q_\beta\right) - g^{\alpha\beta} \nabla_j \nabla_i \left(P_\alpha Q_\beta\right) = 0$$

$$\Rightarrow g^{\alpha\beta} \left[\nabla_i \nabla_j - \nabla_j \nabla_i\right] \left(P_\alpha Q_\beta\right) = 0\, for\ arbitrary P_\alpha\ and\ Q_\beta$$

We have,

$$\nabla_{\mathrm{i}} \nabla_j \left(P_\alpha Q_\beta\right) = \nabla_{\mathrm{i}} \left(P_\alpha \nabla_j Q_\beta + Q_\beta \nabla_j P_\alpha\right)$$

$$= P_\alpha \nabla_{\mathrm{i}} \nabla_j Q_\beta + (\nabla_{\mathrm{i}} P_\alpha)\left(\nabla_j Q_\beta\right) + \left(\nabla_i Q_\beta\right)\left(\nabla_{\mathrm{j}} P_\alpha\right) + Q_\beta \nabla_{\mathrm{i}} \nabla_j P_\alpha$$

$$\nabla_{\mathrm{j}} \nabla_i \left(P_\alpha Q_\beta\right) = \nabla_{\mathrm{j}} \left(P_\alpha \nabla_i Q_\beta + Q_\beta \nabla_i P_\alpha\right)$$

$$= P_\alpha \nabla_j \nabla_i Q_\beta + (\nabla_j P_\alpha)(\nabla_i Q_\beta) + (\nabla_j Q_\beta)(\nabla_i P_\alpha) + Q_\beta \nabla_j \nabla_i P_\alpha$$

$$[\nabla_i \nabla_j - \nabla_j \nabla_i](P_\alpha Q_\beta) = P_\alpha(\nabla_i \nabla_j - \nabla_j \nabla_i)Q_\beta + Q_\beta(\nabla_i \nabla_j - \nabla_j \nabla_i)P_\alpha$$

$$g^{\alpha\beta}[\nabla_i \nabla_j - \nabla_j \nabla_i](P_\alpha Q_\beta) = g^{\alpha\beta} P_\alpha(\nabla_i \nabla_j - \nabla_j \nabla_i)Q_\beta + g^{\alpha\beta} Q_\beta(\nabla_i \nabla_j - \nabla_j \nabla_i)P_\alpha = 0$$

Next we apply the formula[2]

$$[\nabla_i \nabla_j - \nabla_j \nabla_i]A_p = R^n{}_{pij} A_n$$

$$g^{\alpha\beta} P_\alpha R^n{}_{\beta ij} Q_n + g^{\alpha\beta} Q_\beta R^n{}_{\alpha ij} P_n = 0 \; for \; arbitray P_\alpha, Q_\beta$$

$$g^{\alpha\beta} P_\alpha R^0{}_{\beta ij} Q_0 + [g^{\alpha\beta} P_\alpha R^k{}_{\beta ij} Q_k]_{k=1,2,3} + g^{\alpha 0} Q_0 R^n{}_{\alpha ij} P_n + [g^{\alpha k} Q_k R^n{}_{\alpha ij} P_n]_{k=1,2,3} = 0$$

Since P_α and Q_β are arbitrary we make Q_0 five times its previous value

$$5g^{\alpha\beta} P_\alpha R^0{}_{\beta ij} Q_0 + [g^{\alpha\beta} P_\alpha R^k{}_{\beta ij} Q_k]_{k=1,2,3} + 5g^{\alpha 0} Q_0 R^n{}_{\alpha ij} P_n + [g^{\alpha k} Q_k R^n{}_{\alpha ij} P_n]_{k=1,2,3} = 0$$

We could make similar type of adjustments with other components sometimes changing all of them simultaneously in different proportions.

The only solution would be to have$g^{\alpha\beta} = 0$

Conclusion

As mentioned earlier that there is some peculiarity about the covariant operators. Their behavior indicates that the metric coefficients have to disappear

References

1. Spiegel M R, Vector Analysis and an Introduction to Tensor Analysis, Schaum's Outline Series, McGraw-Hill,© 1959, Covariant Derivative, page 197-199
2. Spiegel M R, Vector Analysis and an Introduction to Tensor Analysis, Schaum's Outline Series, McGraw-Hill, ,© 1959, problem76,page 206