

# Proof that there are no odd perfect numbers

Kouji Takaki

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## 1. Abstract

If  $y$  is an odd perfect number, let  $p$  be one of the prime factors of  $y$ , the exponent of  $p$  be an integer  $n(n \geq 1)$ , the prime factors other than  $p$  be  $p_1, p_2, p_3, \dots, p_r$  and the even exponent of  $p_k$  be  $q_k$ .

$$y/p^n = (1 + p + p^2 + \dots + p^n) \prod_{k=1}^r (1 + p_k + p_k^2 + \dots + p_k^{q_k}) / (2p^n) = \prod_{k=1}^r p_k^{q_k}$$

must be satisfied. Let  $m$  be a non negative integer and  $q$  be a positive integer,

$$n = 4m + 1$$

$$p = 4q + 1$$

Letting  $a$ ,  $b$  and  $c$  be odd integers, satisfying following expressions,

$$a = \prod_{k=1}^r (1 + p_k + p_k^2 + \dots + p_k^{q_k})$$

$$b = \prod_{k=1}^r p_k^{q_k}$$

$$c = a/p^n$$

$$2b = c(p^n + \dots + 1)$$

is established. This is a known content. Let  $s(s \geq r)$  be an integer,  $v$  be a rational number,

$$v = \prod_{k=1}^s (1 + p_k + p_k^2 + \dots + p_k^{q_k}) / \prod_{k=1}^s p_k^{q_k}$$

holds. By the consideration of this research paper, since it turned out that if  $v$  is not an integer, by the uniqueness of  $a/b$  there is at most one solution that satisfies this equation for arbitrary  $p$ . Then since by the uniqueness of  $a(p^n + \dots + 1)/(bp^n)$  we proved that there is no solution for  $c = a/p^n$  and  $2b = c(p^n + \dots + 1)$  other than  $(a, b, p, n) = (1, 1, 1, 1)$ , we have obtained a conclusion that there are no odd perfect numbers.

## 2. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself, and the smallest perfect number is

$$1 + 2 + 3 = 6$$

It is 6. Whether an odd perfect number exists or not is currently an unsolved problem in mathematics.

### 3. Proof

Let  $y$  be an odd perfect number, one of the prime factors of  $y$  be odd prime  $p$  and an exponent of  $p$  be an integer  $n(n \geq 1)$ . Let prime factors  $p_1, p_2, p_3, \dots, p_r$  be the odd prime numbers of factors other than  $p$ ,  $q_k$  be the index of  $p_k$ , and an integer  $a$  be a product of series other than prime  $p$ .

$$a = \prod_{k=1}^r (1 + p_k + p_k^2 + \dots + p_k^{q_k}) \dots \textcircled{1}$$

The number of terms  $N$  of variable  $a$  is

$$N = \prod_{k=1}^r (q_k + 1) \dots \textcircled{2}$$

When  $y$  is a perfect number,

$$y = a(1 + p + p^2 + \dots + p^n) - y \quad (n > 0)$$

is established.

$$a \sum_{k=0}^n p^k / 2 = y$$

$$a \sum_{k=0}^n p^k / (2p^n) = y/p^n \dots \textcircled{3}$$

#### 3.1. If $q_k$ has at least one odd integer

Letting the number of terms where  $q_k$  is an odd integer be a positive integer  $u$ , because  $y/p^n = \prod_{k=1}^r p_k^{q_k}$  is an odd integer, the denominator on the left side of the expression  $\textcircled{3}$  has a prime factor 2, from the expression  $\textcircled{2}$  variable  $a$  has more than  $u$  prime factor 2 and variable  $a$  is an even integer. Therefore,  $\sum_{k=0}^n p^k$  must be an odd integer,  $n$  is an even integer and  $u$  is 1.

#### 3.2. When all $q_k$ are even integers

$y/p^n$  is an odd integer, the denominator on the left side of the expression  $\textcircled{3}$  is an even integer, and since  $N$  is an odd integer when  $q_k$  are all even integers, variable  $a$  is an odd integer. Therefore,  $\sum_{k=0}^n p^k$  is necessary to include one prime factor 2,  $\sum_{k=0}^n p^k \equiv 0 \pmod{2}$  is established, and  $n$  must be an odd integer.

From 3.1, 3.2, in order to have an odd perfect number, only one exponent of the prime factor of  $y$  must be an odd integer. We consider the case of 3.2 below.

In order for  $y$  to be an odd perfect number, the following expression must be established.

$$y/p^n = (1 + p + p^2 + \dots + p^n) \prod_{k=1}^r (1 + p_k + p_k^2 + \dots + p_k^{q_k}) / (2p^n) = \prod_{k=1}^r p_k^{q_k}$$

However,  $q_1, q_2, \dots, q_r$  are all even integers.

Here, let  $b$  be an odd integer

$$b = \prod_{k=1}^r p_k^{q_k} \dots \textcircled{4}$$

A following expression is established.

$$y/p^n = a(1 + p + p^2 + \dots + p^n) / (2p^n) = b$$

$$a(p^{n+1} - 1) / (2(p - 1)p^n) = b$$

$$(a - 2b)p^{n+1} + 2bp^n - a = 0 \dots \textcircled{5}$$

Because it is an  $n + 1$  order equation of  $p$ , the solution of the odd prime  $p$  is  $n + 1$  at most for arbitrary  $a$  and  $b$ .

$$(ap - 2bp + 2b)p^n = a$$

Since  $ap - 2bp + 2b$  is an odd integer,  $a/p^n$  is an odd integer. Let  $a/p^n$  be an odd integer  $c$ .

$$ap - 2bp + 2b = c \ (c > 0) \dots \textcircled{6}$$

$$(2b - a)p = 2b - c$$

Since variable  $a$  is an odd integer,  $2b - a$  is an odd integer and  $2b - a \neq 0$

$$p = (2b - c) / (2b - a)$$

Since  $n \geq 1$ ,

$$a - c = cp^n - c \geq cp - c > 0$$

$$a > c$$

is.

From the equation ⑥

$$2b(p - 1) - (ap - c) = 0$$

$$2b - c(p^{n+1} - 1)/(p - 1) = 0$$

$(p^n + \dots + 1)/2$  is an odd integer,  $n = 4m + 1$  must be hold with  $m$  as an integer.

$$2b(p - 1) = c(p^{n+1} - 1)$$

$$2b = c(p^n + \dots + 1)$$

$$2b = c(p + 1)(p^{n-1} + p^{n-3} + \dots + 1) \dots \textcircled{7}$$

Since  $b$  is an odd integer when  $p + 1$  is not a multiple of 4,  $p - 1$  must be a multiple of 4. A positive integer is taken as  $q$ .

$$p = 4q + 1$$

is established.

When  $p > 1$

$$p^n - 1 < p^n$$

$$(p^n - 1)/(p - 1) < p^n/(p - 1)$$

$$p^{n-1} + \dots + 1 < p^n/(p - 1)$$

Since  $p$  is an odd prime number satisfying  $p = 4q + 1$  and  $p \geq 5$ ,

$$p^{n-1} + \dots + 1 < p^n/4$$

$$2b - a = c(p^n + \dots + 1) - cp^n = c(p^{n-1} + \dots + 1)$$

$$2b - a < cp^n/4 = a/4$$

$$2b < 5a/4$$

$$a > 8b/5 \dots \textcircled{8}$$

Let  $a_k$  and  $b_k$  be odd integers and if

$$a_k = 1 + p_k + p_k^2 + \dots + p_k^{q_k}, \quad b_k = p_k^{q_k},$$

$$a_k - b_k < b_k/(p_k - 1)$$

$$a_k < b_k p_k / (p_k - 1)$$

$$a = \prod_{k=1}^r a_k < \prod_{k=1}^r b_k p_k / (p_k - 1) = b \prod_{k=1}^r p_k / (p_k - 1)$$

$$a/b < \prod_{k=1}^r p_k / (p_k - 1)$$

When  $r = 1$ , since  $a/b < 3/2$  is established, it becomes inappropriate contrary to inequality ⑧.

From the expression ⑦,

$$b = c(p + 1)/2 \times (p^{n-1} + p^{n-3} + \dots + 1)$$

holds. Since  $(p + 1)/2$  is the product of only prime numbers of  $b$ , let  $d_k$  be the index,

$$(p + 1)/2 = \prod_{k=1}^r p_k^{d_k}$$

$$p = 2 \prod_{k=1}^r p_k^{d_k} - 1 \dots \text{⑨}$$

From  $a = cp^n$  and the expression ⑦,

$$2bp^n = a(p^n + \dots + 1)$$

$$a(p^n + \dots + 1)/(2bp^n) = 1 \dots \text{(A)}$$

When  $r = 1$ ,

$$a = (p_1^{q_1+1} - 1)/(p_1 - 1)$$

$$b = p_1^{q_1}$$

The equation (A) does not hold since there is no odd perfect number when  $r = 1$ .

Let R be a rational number,

$$R = a(p^n + \dots + 1)/(2bp^n)$$

Let b' be a rational number and let  $A_k$  and  $B_k$  to be odd integers,

$$b' = (p_k^{q_k+1} - 1)/(p_k^{q_k}(p_k - 1)) > 1$$

$$A_k = (p_k^{q_k+1} - 1)/(p_k - 1)$$

$$B_k = p_k^{q_k}$$

Multiplying R by b', there are both cases that  $p_k$  increases p or does not change.

When multiplied by b', the rate of change of R is  $A_{r+1}p^n(p'^n + \dots + 1)/(B_{r+1}p^n(p^n + \dots + 1))$ , if p after variation is p'. If the rate of change of R is 1,

$$A_{r+1}p^n(p'^n + \dots + 1)/(B_{r+1}p^n(p^n + \dots + 1)) = 1$$

$$A_{r+1}p^n(p'^n + \dots + 1) = B_{r+1}p^n(p^n + \dots + 1)$$

This expression does not hold since the right side is not a multiple of p when  $p' > p$ , and  $A_{r+1} > B_{r+1}$  holds when  $p' = p$ . Due to this operation, R may be larger or smaller than the original value since the rate of change of R does not become 1.

Assuming that  $R = 1$  in some r, letting x be an integer and by multiplying fractions

$b' = A_{r+1}/B_{r+1}$ ,  $b'' = A_{r+2}/B_{r+2}$ ,  $\dots b'''\dots' = A_x/B_x$  to R. Furthermore, assuming that

$A_{t+1}A_{t+2} \dots A_r$  is not a multiple of p, R is divided by  $A_{t+1}/B_{t+1}$ ,  $A_{t+2}/B_{t+2}, \dots A_r/B_r$

and it is assumed that finally  $R = 1$ . At this time, assuming that n changes to  $n_{r+1}$ ,

the change rate of R by this operation when multiplying by  $A_{r+1}/B_{r+1}$  is

$$A_{r+1}p^n(p^{n_{r+1}} + \dots + 1)/(B_{r+1}p^{n_{r+1}}(p^n + \dots + 1))$$

$$\begin{aligned} 1 \times B_{t+1}p^n(p^{n_{t+1}} + \dots + 1)/(A_{t+1}p^{n_{t+1}}(p^n + \dots + 1)) \times \dots \times B_r p^{n_{r-1}}(p^{n_r} + \dots \\ + 1)/(A_r p^{n_r}(p^{n_{r-1}} + \dots + 1)) \times A_{r+1}p^{n_r}(p^{n_{r+1}} + \dots + 1)/(B_{r+1}p^{n_{r+1}}(p^{n_r} \\ + \dots + 1)) \times A_{r+2}p^{n_{r+1}}(p^{n_{r+2}} + \dots + 1)/(B_{r+2}p^{n_{r+2}}(p^{n_{r+1}} + \dots + 1)) \times \dots \\ \times A_x p^{n_x-1}(p^{n_x} + \dots + 1)/(B_x p^{n_x}(p^{n_x-1} + \dots + 1)) = 1 \end{aligned}$$

$$B_{t+1}B_{t+2} \dots B_r A_{r+1}A_{r+2} \dots A_x p^{n-n_x}(p^{n_x} + \dots + 1)$$

$$= A_{t+1}A_{t+2} \dots A_r B_{r+1}B_{r+2} \dots B_x (p^n + \dots + 1) \dots (B)$$

When  $n_x < n$ , it becomes contradiction since the right side of above expression does not include the prime factor p.

When  $n_x = n$ ,

$$B_{t+1}B_{t+2} \dots B_r A_{r+1}A_{r+2} \dots A_x = A_{t+1}A_{t+2} \dots A_r B_{r+1}B_{r+2} \dots B_x \dots (C)$$

Let  $s(s \geq r)$  be an integer and  $v$  be a rational number, if

$$v = \prod_{k=1}^s (1 + p_k + p_k^2 + \dots + p_k^{q_k}) / \prod_{k=1}^s p_k^{q_k}$$

holds, assume that  $v$  is not an integer.  $\dots(D)$

Let  $e_r, f_r$  be odd integers and  $g_r$  be a rational number,

$$e_r = \prod_{k=1}^r (p_k^{q_k} + \dots + 1), f_r = \prod_{k=1}^r p_k^{q_k}, g_r = e_r/f_r$$

holds.

$$g_{r+1} = e_{r+1}/f_{r+1} = e_r/f_r \times (p_{r+1}^{q_{r+1}} + \dots + 1)/p_{r+1}^{q_{r+1}} > e_r/f_r = g_r$$

Let  $q_1'$  be an even integer and  $q_1' > q_1$  holds. Let  $g_r$  be  $g_r'$  when  $q_1$  becomes  $q_1'$ ,

$$g_r' = (p_1^{q_1}(p_1^{q_1'} + \dots + 1)/p_1^{q_1'}(p_1^{q_1} + \dots + 1))g_r > g_r$$

is established.

It is assumed that  $q_k$  becomes  $q_k - h_k$  by changing  $q_k$  than before for  $g_r$ .  $h_k$  is an even integer. Then assume that  $r$  becomes  $s(s > r)$ ,  $g_s = g_r$  and  $g_s$  is not changed.

$$g_s/g_r = p_{r+1}^{q_{r+1}} \times \dots \times p_s^{q_s} / ((p_{r+1}^{q_{r+1}} + \dots + 1) \times \dots \times (p_s^{q_s} + \dots + 1)) \times p_1^{q_1} \times \dots$$

$$\times p_r^{q_r} (p_1^{q_1 - h_1} + \dots + 1) \dots (p_r^{q_r - h_r} + \dots + 1) / (p_1^{q_1 - h_1} \times \dots$$

$$\times p_r^{q_r - h_r} (p_1^{q_1} + \dots + 1) \dots (p_r^{q_r} + \dots + 1)) = 1$$

$$p_{r+1}^{q_{r+1}} \times \dots \times p_s^{q_s} / ((p_{r+1}^{q_{r+1}} + \dots + 1) \times \dots \times (p_s^{q_s} + \dots + 1)) \times p_1^{h_1} \times \dots$$

$$\times p_r^{h_r} (p_1^{q_1 - h_1} + \dots + 1) \dots (p_r^{q_r - h_r} + \dots$$

$$+ 1) / ((p_1^{q_1} + \dots + 1) \dots (p_r^{q_r} + \dots + 1)) = 1$$

$$p_{r+1}^{q_{r+1}} \times \dots \times p_s^{q_s} \times p_1^{h_1} \times \dots \times p_r^{h_r} (p_1^{q_1 - h_1} + \dots + 1) \dots (p_r^{q_r - h_r} + \dots + 1)$$

$$= (p_1^{q_1} + \dots + 1) \dots (p_r^{q_r} + \dots + 1) (p_{r+1}^{q_{r+1}} + \dots + 1) \dots (p_s^{q_s} + \dots + 1)$$

When  $h_k < 0 (1 \leq k \leq r)$ , multiply both sides by  $p_k^{-h_k}$  so that both sides become integers. When  $\prod_{k=r+1}^s (A_k/B_k)$  is not an integer, if both sides are divided by the prime numbers from  $p_{r+1}$  to  $p_s$ , at least one prime number among the prime numbers from  $p_{r+1}$  to  $p_s$  are left on the left side. Since  $a = \prod_{k=1}^r A_k = cp^n$  holds and from the expression ⑦,  $c$  must be a product of primes from  $p_1$  to  $p_r$ . Thereby, the above equation does not hold, since it is inappropriate when there is even one prime number other than  $p_1$  to  $p_r$  and  $p$ . When changing the value of  $p_k$ , it is equivalent to dividing by  $p_k^{q_k}$  and then multiplying by new  $p_k^{q_k}$ , so it is sufficient to consider only the changes of  $q_k$  and  $r$ . From above, since  $g_r$  does not chord the original value when  $q_k$  or  $r$  is increased or decreased, it takes unique values for the variables  $p_k, q_k, r$ .



From the above proof,

$$g_r = A_1 A_2 \dots A_s / B_1 B_2 \dots B_s \times A_{r+1} A_{r+2} \dots A_x / B_{r+1} B_{r+2} \dots B_x$$

$g_r$  must be represented uniquely, and the expression (C) does not satisfied. When dividing by the prime number in the expression ⑨, a contradiction arises since the prime number not included in  $b$  is in the expression ⑨. Therefore, when  $\prod_{k=r+1}^s (A_k / B_k)$  is not an integer and  $p$  holds  $p \equiv 1 \pmod{4}$  and  $p \geq 5$ , the number of the solution  $(a, b, p, n)$  satisfying  $R = 1$  is at most one.

Since  $(a, b, p, n) = (1, 1, 1, 1)$  is inappropriate solution for the equation (A). At this time, since  $a = b = 1$  and  $r = 0$  that  $\prod_{k=r+1}^s (A_k / B_k)$  is not an integer is same that the condition (D) holds, and since the expression (C) becomes contradiction, there is one inappropriate solution when  $n_x = n = 1$ . Therefore, if the condition (D) holds, there are no odd perfect numbers when  $n = 1$ .

Define the operation [multiplication] and the operation [division] as follows.

Assuming that  $p$  in the equation of  $R$  is replaced by  $p'$  by multiplying  $A_i / B_j$ , define operation [multiplication] to  $R$  as follows.

$$p' = 2 \prod_{k=1}^r p_k^{d_k} \times p_i^{d_i} - 1$$

$$0 \leq d_i \leq q_i$$

Here, let  $i$  be  $i > r$ . Suppose operation [division] is division by  $A_j / B_j$  for  $R$ , and if  $p_j$  is included in  $p$  in the expression  $R$ ,  $p_j$  is deleted as  $d_j = 0$ . Here, assuming that  $j$  satisfies  $1 \leq j \leq r$ .

In the proof of the expression (B), it is assumed that  $p$  changes on the way, and finally  $p$  becomes  $p_x$ .

$$A_1 \dots A_r = c p^n$$

$$2B_1 \dots B_r = c(p^n + \dots + 1)$$

$$A_1 \dots A_x = c' p_x^{n_x}$$

$$2B_1 \dots B_x = c'(p_x^{n_x} + \dots + 1)$$

It is assumed that the above expressions are satisfied.

$$\begin{aligned} B_{t+1} B_{t+2} \dots B_r A_{r+1} A_{r+2} \dots A_x p^n (p_x^{n_x} + \dots + 1) \\ = A_{t+1} A_{t+2} \dots A_r B_{r+1} B_{r+2} \dots B_x p_x^{n_x} (p^n + \dots + 1) \end{aligned}$$

$$\begin{aligned} B_{t+1} B_{t+2} \dots B_r A_1 \dots A_r A_{r+1} A_{r+2} \dots A_x p^n (p_x^{n_x} + \dots + 1) \\ = A_1 \dots A_r A_{t+1} A_{t+2} \dots A_r B_{r+1} B_{r+2} \dots B_x p_x^{n_x} (p^n + \dots + 1) \end{aligned}$$

$$\begin{aligned}
& B_{t+1}B_{t+2} \dots B_r c' p_x^{n_x} p^n (p_x^{n_x} + \dots + 1) \\
& \quad = A_1 \dots A_r A_{t+1} A_{t+2} \dots A_r B_{r+1} B_{r+2} \dots B_x p_x^{n_x} (p^n + \dots + 1) \\
& B_{t+1}B_{t+2} \dots B_r c' p^n (p_x^{n_x} + \dots + 1) = A_1 \dots A_r A_{t+1} A_{t+2} \dots A_r B_{r+1} B_{r+2} \dots B_x (p^n + \dots + 1) \\
& B_1 \dots B_r B_{t+1} B_{t+2} \dots B_r c' p^n (p_x^{n_x} + \dots + 1) \\
& \quad = A_1 \dots A_r A_{t+1} A_{t+2} \dots A_r B_1 \dots B_r B_{r+1} B_{r+2} \dots B_x (p^n + \dots + 1) \\
& B_1 \dots B_r B_{t+1} B_{t+2} \dots B_r c' p^n (p_x^{n_x} + \dots + 1) \\
& \quad = A_1 \dots A_r A_{t+1} A_{t+2} \dots A_r c' (p_x^{n_x} + \dots + 1) / 2 \times (p^n + \dots + 1) \\
& B_1 \dots B_r B_{t+1} B_{t+2} \dots B_r p^n = A_1 \dots A_r A_{t+1} A_{t+2} \dots A_r / 2 \times (p^n + \dots + 1)
\end{aligned}$$

$$c(p^n + \dots + 1) / 2 \times B_{t+1} B_{t+2} \dots B_r p^n = c p^n A_{t+1} A_{t+2} \dots A_r / 2 \times (p^n + \dots + 1)$$

$$B_{t+1} B_{t+2} \dots B_r = A_{t+1} A_{t+2} \dots A_r$$

is established. It becomes contradiction since  $A_k > B_k$  holds when the operations [division] are performed.

Consider a tree whose vertex is  $(a, b, p, n) = (1, 1, 1, 1)$ , and when the operations [multiplication] are performed, it becomes a child node. For example, consider a child node connected to a vertex as follows.

$$(a, b, p, n) = (13, 9, 5, 5) \text{ as } p_1 = 3, q_1 = 2 \text{ and } d_1 = 1$$

$$(a, b, p, n) = (13, 9, 17, 9) \text{ as } p_1 = 3, q_1 = 2 \text{ and } d_1 = 2$$

$$(a, b, p, n) = (57, 49, 97, 13) \text{ as } p_1 = 7, q_1 = 2 \text{ and } d_1 = 2$$

Suppose that the operations [multiplication] for changing the value of  $p$  are performed first, and then the operations [multiplication] for not changing the value of  $p$  are performed to create a tree structure. Here, when there is a solution in a certain  $p$  and there is a solution even in the other value  $p'$ , considering a set of line segments connecting these two points in four-dimensional space  $(a, b, p, n)$ . If  $R = 1$  holds again when performing operation [multiplication] from one point where  $R = 1$ ,

$$\begin{aligned}
& 1 \times A_{r+1} p^n (p_{r+1}^{n_{r+1}} + \dots + 1) / (B_{r+1} p_{r+1}^{n_{r+1}} (p^n + \dots + 1)) \times A_{r+2} p_{r+1}^{n_{r+1}} (p_{r+2}^{n_{r+2}} + \dots \\
& \quad + 1) / (B_{r+2} p_{r+2}^{n_{r+2}} (p_{r+1}^{n_{r+1}} + \dots + 1)) \times \dots \times A_x p_{x-1}^{n_{x-1}} (p_x^{n_x} + \dots \\
& \quad + 1) / (B_x p_x^{n_x} (p_{x-1}^{n_{x-1}} + \dots + 1)) = 1
\end{aligned}$$

$$A_{r+1} A_{r+2} \dots A_x / (B_{r+1} B_{r+2} \dots B_x) = p_x^{n_x} (p^n + \dots + 1) / (p^n (p_x^{n_x} + \dots + 1))$$

$$A_1 A_2 \dots A_x (p_x^{n_x} + \dots + 1) / (B_1 B_2 \dots B_x p_x^{n_x}) = A_1 A_2 \dots A_r (p^n + \dots + 1) / (B_1 B_2 \dots B_r p^n) \dots (E)$$

Assume that  $G_r = A_1 A_2 \dots A_r (p^n + \dots + 1) / (B_1 B_2 \dots B_x p^n)$  holds. Here, it is assumed that  $q_k$  becomes  $q_k - h_k$  by changing  $q_k$  than before and  $n$  becomes  $n - h$  ( $n - h > 0$ ) for  $G_r$ .  $h_k$  is an even integer and  $h$  is a non-negative integer that is a multiple of 4. Then assuming that  $r$  becomes  $s$  ( $s > r$ ),  $G_s = G_r$  and  $G_s$  is not changed, by the same calculation of  $g_s/g_r$ ,

$$\begin{aligned} G_s/G_r &= p_{r+1}^{q_{r+1}} \times \dots \times p_s^{q_s} / ((p_{r+1}^{q_{r+1}} + \dots + 1) \times \dots \times (p_s^{q_s} + \dots + 1)) \times p_1^{q_1} \times p_2^{q_2} \times \dots \\ &\quad \times p_r^{q_r} p^n (p_1^{q_1 - h_1} + \dots + 1) \dots (p_r^{q_r - h_r} + \dots + 1) (p^{n-h} + \dots + 1) / (p_1^{q_1 - h_1} \\ &\quad \times \dots \times p_r^{q_r - h_r} p^{n-h} (p_1^{q_1} + \dots + 1) \dots (p_r^{q_r} + \dots + 1) (p^n + \dots + 1)) = 1 \\ p_{r+1}^{q_{r+1}} \times \dots \times p_s^{q_s} p_1^{q_1} \times \dots \times p_r^{h_r} (p_1^{q_1 - h_1} + \dots + 1) \dots (p_r^{q_r - h_r} + \dots + 1) (p^n + \dots + p^h) \\ &= (p_1^{q_1} + \dots + 1) \dots (p_r^{q_r} + \dots + 1) (p^n + \dots + 1) (p_{r+1}^{q_{r+1}} + \dots + 1) \dots (p_s^{q_s} \\ &\quad + \dots + 1) \end{aligned}$$

Since  $\prod_{k=1}^r A_k = cp^n$  holds,

$$\begin{aligned} p_{r+1}^{q_{r+1}} \times \dots \times p_s^{q_s} p_1^{q_1} \times \dots \times p_r^{h_r} (p_1^{q_1 - h_1} + \dots + 1) \dots (p_r^{q_r - h_r} + \dots + 1) (p^{n-h} + \dots + 1) \\ = cp^{n-h} (p^n + \dots + 1) (p_{r+1}^{q_{r+1}} + \dots + 1) \dots (p_s^{q_s} + \dots + 1) \end{aligned}$$

When  $h_k < 0$  ( $1 \leq k \leq r$ ), multiply both sides by  $p_k^{-h_k}$  so that both sides become integers. When  $\prod_{k=r+1}^s (A_k/B_k)$  is not an integer, if both sides are divided by the prime numbers from  $p_{r+1}$  to  $p_s$ , at least one prime number among the prime numbers from  $p_{r+1}$  to  $p_s$  are left on the left side. Because  $c$  and  $p^n + \dots + 1$  are products of prime numbers from  $p_1$  to  $p_r$  and in the case of  $s > r + 1$ , the left side has prime numbers that is not on the right side as a factor, this expression does not hold. In the case of  $s = r + 1$ , when  $p \neq p_s$ , this expression does not hold in the same way. When  $p = p_s$  and  $q_s > n - h$ , since there is a prime factor  $p$  only on the left side, this expression does not hold. Therefore, since except for the case of  $s = r + 1$ ,  $p = p_s$  and  $q_s < n - h$   $G_r$  must be uniquely expressed, the expression (E) does not hold. When  $s = r + 1$ ,  $p = p_s$  and  $q_s < n - h$ , substituting  $B_x = p^{q_s}$  into the expression (E) as  $x = r + 1$ ,

$$\begin{aligned} A_1 A_2 \dots A_r (p^{q_s} + \dots + 1) (p_x^{n_x} + \dots + 1) / (B_1 B_2 \dots B_r p^{q_s} p_x^{n_x}) \\ = A_1 A_2 \dots A_r (p^n + \dots + 1) / (B_1 B_2 \dots B_r p^n) \end{aligned}$$

$$(p^{q_s} + \dots + 1) (p_x^{n_x} + \dots + 1) / (p^{q_s} p_x^{n_x}) = (p^n + \dots + 1) / p^n$$

$$(p^{q_s} + \dots + 1) (p_x^{n_x} + \dots + 1) p^{n - q_s} = (p^n + \dots + 1) p_x^{n_x}$$

Since the right side does not have a prime number  $p$  as a factor, this expression does not hold.

When one point is  $(a, b, p, n) = (1, 1, 1, 1)$ , since  $r = 0$ , that  $\prod_{k=r+1}^s (A_k/B_k)$  is not an integer is same that the condition (D) holds. If the condition (D) holds, when  $s > r + 1$  or  $p \neq p_s$ ,  $G_s \neq G_r$  holds similarly and when  $s = r + 1$  and  $p = p_s$  it becomes inappropriate, since prime number  $p_s$  is 1.

If the condition (D) does not hold,  $v = a/b$  when  $s = r$ . Because the equation (A) must be satisfied at a point other than the point  $(a, b, p, n) = (1, 1, 1, 1)$ , considering  $v$  becomes an integer,

$$v = a/b = 2p^n/(p^n + \dots + 1)$$

$$2p^n = v(p^n + \dots + 1)$$

Let  $w$  be an integer and if  $v = wp^n$  holds,

$$2 = w(p^n + \dots + 1)$$

When  $p \equiv 1 \pmod{4}$ ,  $p \geq 5$  and  $n \equiv 1 \pmod{4}$ ,  $n \geq 1$ ,

$$p^n + \dots + 1 \geq 6$$

At this time, it becomes inappropriate, since  $w$  is not an integer. Therefore, except for  $(a, b, p, n) = (1, 1, 1, 1)$ , there is no solution satisfying the equation (A). From the above, there are no odd perfect numbers.

4. Complement

From the equation ⑤,

$$2bp^n(p-1) = a(p^{n+1} - 1)$$

$$2 = a(p^{n+1} - 1)/(bp^n(p-1))$$

$$2 = (p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1) \dots (p_r^{q_r+1} - 1)(p^{n+1} - 1)$$

$$/(p_1^{q_1} p_2^{q_2} \dots p_r^{q_r} p^n (p_1 - 1)(p_2 - 1) \dots (p_r - 1)(p - 1))$$

$$2(p_1^{q_1+1} - p_1^{q_1})(p_2^{q_2+1} - p_2^{q_2}) \dots (p_r^{q_r+1} - p_r^{q_r})(p^{n+1} - p^n)$$

$$= (p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1) \dots (p_r^{q_r+1} - 1)(p^{n+1} - 1)$$

We consider when  $r = 2$ .

$$(p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1)(p^{n+1} - 1) = 2(p_1^{q_1+1} - p_1^{q_1})(p_2^{q_2+1} - p_2^{q_2})(p^{n+1} - p^n)$$

Let  $s, t, u$  be integers,

$$s = p_1^{q_1+1} - 1$$

$$t = p_2^{q_2+1} - 1$$

$$u = p^{n+1} - 1$$

are.

$$stu = 2(p_1^{q_1+1} - 1 - (p_1^{q_1} - 1))(p_2^{q_2+1} - 1 - (p_2^{q_2} - 1))(p^{n+1} - 1 - (p^n - 1))$$

$$stu = 2(s - (s+1)/p_1 + 1)(t - (t+1)/p_2 + 1)(u - (u+1)/p + 1)$$

$$pp_1 p_2 stu = 2((s+1)p_1 - (s+1))((t+1)p_2 + (t+1))((u+1)p + (u+1))$$

$$pp_1 p_2 stu = 2(s+1)(p_1 - 1)(t+1)(p_2 - 1)(u+1)(p - 1)$$

$$stu/((s+1)(t+1)(u+1)) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$$

Since  $stu/((s+1)(t+1)(u+1))$  is a monotonically increasing function for variables  $s, t$  and  $u$ , if

$$s \geq 3^{2+1} - 1 = 26, p_1 = 3, q_1 = 2$$

$$t \geq 7^{2+1} - 1 = 342, p_2 = 7, q_2 = 2$$

$$u \geq 5^2 - 1 = 24, p = 5, n = 1$$

holds,

$$stu/((s+1)(t+1)(u+1)) \geq 26 \times 342 \times 24 / (27 \times 343 \times 25) = 7904/8575$$

$$2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) = 2 \times 2 \times 6 \times 4 / (3 \times 7 \times 5) = 32/35$$

Since  $stu/((s+1)(t+1)(u+1))$  is limited to 1 when  $s, t$  and  $u$  are infinite,  $stu/((s+1)(t+1)(u+1)) < 1$

If  $f(p_1, p_2, p) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$  holds, it is sufficient to consider a combination where  $f(p_1, p_2, p) < 1$ .

$$f(3,7,5) = 2 \times 2 \times 6 \times 4 / (3 \times 7 \times 5) = 32/35$$

$$f(3,11,5) = 2 \times 2 \times 10 \times 4 / (3 \times 11 \times 5) = 32/33$$

$$f(3,13,5) = 2 \times 2 \times 12 \times 4 / (3 \times 13 \times 5) = 64/65$$

$$f(3,17,5) = 2 \times 2 \times 16 \times 4 / (3 \times 17 \times 5) = 256/255$$

$$f(3,7,13) = 2 \times 2 \times 6 \times 12 / (3 \times 7 \times 13) = 96/91$$

$$f(3,5,17) = 2 \times 2 \times 4 \times 16 / (3 \times 5 \times 17) = 256/255$$

From the above, when  $r = 2$ , a combination  $(p_1, p_2, p) = (3,7,5), (3,11,5), (3,13,5)$  can be considered.

Let  $q_k$  be 2 and  $n = 1$ , if  $g(p_1, p_2, p) = (p_1^3 - 1)(p_2^3 - 1)(p^2 - 1)/(p_1^3 p_2^3 p^2)$ ,

$$g(3,7,5) = 26 \times 342 \times 24 / (3^3 7^3 5^2) = 7904/8575 > 32/35$$

$$g(3,11,5) = 26 \times 1330 \times 24 / (3^3 11^3 5^2) = 55328/59895$$

$$g(3,13,5) = 26 \times 2196 \times 24 / (3^3 13^3 5^2) = 3904/4225$$

Since the function  $g$  is the minimum in the case of  $q_k = 2$  and  $n = 1$ , there is no solution  $q_k$  and  $n$  when  $g > f$ , so the case of  $(p_1, p_2, p) = (3,7,5)$  becomes unsuitable.

$$stu / ((s + 1)(t + 1)(u + 1)) = 2(p_1 - 1)(p_2 - 1)(p - 1) / (p_1 p_2 p)$$

$$(p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1)(p^{n+1} - 1) / (p_1^{q_1+1} p_2^{q_2+1} p^{n+1})$$

$$= 2(p_1 - 1)(p_2 - 1)(p - 1) / (p_1 p_2 p)$$

If  $F(p_1, p_2, p) = (p_1 - 1)(p_2 - 1)(p - 1) / (p_1 p_2 p)$ ,

$$F(p_1^{q_1+1}, p_2^{q_2+1}, p^{n+1}) = 2F(p_1, p_2, p)$$

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## 6. References

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