Proof that there are no odd perfect numbers

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1. Abstract

For $y$ to be a perfect number, if one of the prime factors is $p$, the exponent of $p$ is an integer $n(n \geq 1)$, the prime factors other than $p$ are $p_1, p_2, p_3, \cdots p_r$ and the even exponent of $p_k$ is $q_k$,

$$y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k})/(2p^n) = \prod_{k=1}^{r} p_k^{q_k}$$

must be satisfied. Let $m$ be non-negative integer and $q$ be positive integer,

$$n = 4m + 1$$
$$p = 4q + 1$$

Letting $b$ and $c$ be odd integers, satisfying following expressions,

$$b = \prod_{k=1}^{r} p_k^{q_k}$$
$$c = \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k})/p^n$$

$$2b = c(p^n + \cdots + 1)$$

is established. This is a known content. By the consideration of this research paper, since it turned out that there is at most one solution that satisfies this equation for $p$, and $p$ is unique in the range of $p \geq 5$, we have obtained the conclusion that there are no odd perfect numbers when $n = 1$ and the number is one at most when $n \geq 5$.

2. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself, and the smallest perfect number is

$$1 + 2 + 3 = 6$$

It is 6. Whether an odd perfect number exists or not is currently an unsolved problem.
3. Proof

An odd perfect number is \( y \), one of them is an odd prime number \( p \), an exponent of \( p \) is an integer \( n \) \((n \geq 1)\). Let \( p_1, p_2, p_3, \ldots, p_r \) be the odd prime numbers of factors other than \( p \), \( q_k \) the index of \( p_k \), and variable \( a \) be the sum of product combinations other than prime \( p \).

\[
a = \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k}) \quad \text{①}
\]

The number of terms \( N \) of variable \( a \) is

\[
N = \prod_{k=1}^{r} (q_k + 1) \quad \text{②}
\]

When \( y \) is a perfect number,

\[
y = a(1 + p + p^2 + \cdots + p^n) - y \quad (n > 0)
\]

is established.

\[
a \sum_{k=0}^{n} p^k / 2 = y
\]

\[
a \sum_{k=0}^{n} p^k / (2p^n) = y / p^n \quad \text{③}
\]

3.1. If \( q_k \) has at least one odd integer

Letting the number of terms where \( q_k \) is an odd integer be a positive integer \( u \), because \( y / p^n = \prod_{k=1}^{r} p_k^{q_k} \) is an odd integer, the denominator on the left side of the expression ③ has a prime factor 2, from the expression ② variable \( a \) has more than \( u \) prime factor 2 and variable \( a \) is an even integer. Therefore \( \sum_{k=0}^{n} p^k \) must be an odd integer, \( n \) is an even integer and \( u \) is 1.

3.2. When all \( q_k \) are even integers

\( y / p^n \) is an odd integer, the denominator on the left side of the expression ③ is an even integer, and since \( N \) is an odd integer when \( q_k \) are all even integers, variable \( a \) is an odd integer. Therefore \( \sum_{k=0}^{n} p^k \) is necessary to include one prime factor 2, \( \sum_{k=0}^{n} p^k \equiv 0 \pmod{2} \) is established, and \( n \) must be an odd integer.

From 3.1, 3.2, in order to have an odd perfect number, only one exponent of the prime factor of \( y \) must be an odd integer and variable \( a \) must be an odd integer. We consider the case of 3.2 below.
In order for $y$ to be a perfect number, the following expression must be established.

$$\frac{y}{p^n} = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} \left(1 + p_k + p_k^2 + \cdots + p_k^{q_k}\right) / (2p^n) = \prod_{k=1}^{r} p_k^{q_k}$$

However, $q_1, q_2, \ldots, q_r$ are all even integers.

Here, let $b$ be an integer

$$b = \prod_{k=1}^{r} p_k^{q_k} \quad \text{(4)}$$

A following expression is established.

$$\frac{y}{p^n} = a(1 + p + p^2 + \cdots + p^n)/(2p^n) = b$$

$$a(p^{n+1} - 1)/(2(p - 1)p^n) = b$$

$$(a - 2b)p^{n+1} + 2bp^n - a = 0 \quad \text{(5)}$$

Because it is an $n+1$ order equation of $p$, the solution of the odd prime $p$ is $n+1$ at most.

$$(ap - 2bp + 2b)p^n = a$$

Since $ap - 2bp + 2b$ is an odd integer, $a/p^n$ is an odd integer, which is $c$.

$$ap - 2bp + 2b = c (c > 0) \quad \text{(6)}$$

$$2b - a)p = 2b - c$$

Since variable $a$ is an odd integer, $2b - a$ is an odd integer and $2b - a \neq 0$

$$p = (2b - c)/(2b - a)$$
Since $n \geq 1$

$a - c = cp^n - c \geq cp - c > 0$

$a > c$

is.

From the equation ⑥

$2b(p - 1) - (ap - c) = 0$

$2b - c(p^{n+1} - 1)/(p - 1) = 0$

$(p^n + \cdots + 1)/2$ is an odd integer, $n = 4m + 1$ is required with $m$ as an integer.

$2b(p - 1) = c(p^{n+1} - 1)$

$2b = c(p^n + \cdots + 1)$

$2b = c(p + 1)(p^{n-1} + p^{n-3} + \cdots + 1)$ ...

$b$ is an odd integer when $p + 1$ is not a multiple of 4. It is necessary that $p - 1$ be a multiple of 4. A positive integer is taken as $q$.

$p = 4q + 1$

is established.

When $p > 1$

$p^n - 1 < p^n$

$(p^n - 1)/(p - 1) < p^n/(p - 1)$

$p^{n-1} + \cdots + 1 < p^n/(p - 1)$ ...

Since $p$ is an odd prime number satisfying $p = 4q + 1$ and $p \geq 5$

$p^{n-1} + \cdots + 1 < p^n/4$

$2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1)$

$2b - a < cp^n/4 = a/4$

$2b < 5a/4$

$a > 8b/5$ ...
Let $a_k$ and $b_k$ be integers and if

\[a_k = 1 + p_k + p_k^2 + \cdots + p_k^{q_k}, \quad b_k = p_k^{q_k},\]

\[a_k - b_k < b_k/(p_k - 1),\]

\[a_k < b_k p_k/(p_k - 1)\]

\[a = \prod_{k=1}^{r} a_k < \prod_{k=1}^{r} b_k p_k/(p_k - 1) = b \prod_{k=1}^{r} p_k/(p_k - 1)\]

\[a/b < \prod_{k=1}^{r} p_k/(p_k - 1)\]

When $r = 1$, since $a/b < 3/2$ is established, it becomes inappropriate contrary to inequality ⑨.

From the expression ⑦,

\[b = c(p + 1)/2 \times (p^{n-1} + p^{n-3} + \cdots + 1)\]

holds. Since $(p + 1)/2$ is the product of only prime numbers of $b$, let $d_k$ be the index,

\[(p + 1)/2 = \prod_{k=1}^{r} p_k^{d_k}\]

\[p = 2 \prod_{k=1}^{r} p_k^{d_k} - 1\]

From $a = cp^n$ and the expression ⑦,

\[2bp^n = a(p^n + \cdots + 1)\]

\[a(p^n + \cdots + 1)/(2bp^n) = 1 \ldots (A)\]

When $r = 1$,

\[a = (p_1^{q_1+1} - 1)/(p_1 - 1)\]

\[b = p_1^{q_1}\]

The equation (A) does not hold since there is no odd perfect number when $r = 1$. 
Let $R$ be a rational number,
$$R = a(p^n + \cdots + 1)/(2bp^n)$$
Let $b'$ be a rational number and let $A$ and $B$ to be an integer,
$$b' = (p_kq_k+1 - 1)/(p_kq_k(p_k - 1)) > 1$$
$$A_k = (p_kq_k+1 - 1)/(p_k - 1)$$
$$B_k = p_kq_k$$

Multiplying $R$ by $b'$, there are both cases that $p_k$ increases $p$ or does not change. When multiplied by $b'$, the rate of change of $R$ is $A_{r+1}p^n(p^n + \cdots + 1)/(B_{r+1}p^n(p^n + \cdots + 1))$, if $p$ after variation is $p'$. If the rate of change of $R$ is 1,
$$A_{r+1}p^n(p^n + \cdots + 1)/(B_{r+1}p^n(p^n + \cdots + 1)) = 1$$

This expression does not hold since the right side is not a multiple of $p$ when $p' > p$, and $A_{r+1} > B_{r+1}$ holds when $p' = p$. Due to this operation, $R$ may be larger or smaller than the original value since the rate of change of $R$ does not become 1.

Assuming that $R = 1$ in some $r$, letting $x$ be an integer and by multiplying fractions $b' = A_{r+1}/B_{r+1}$, $b'' = A_{r+2}/B_{r+2}$, \ldots $b'''' = A_x/B_x$ to $R$. Furthermore, assuming that $A_{s+1}A_{s+2} \ldots A_r$ is not a multiple of $p$, $R$ is divided by $A_{s+1}/B_{s+1}, A_{s+2}/B_{s+2}, \ldots A_r/B_r$ and it is assumed that finally $R = 1$. At this time, assuming that $n$ changes, the change rate of $R$ by this operation when multiplying by $A_{r+1}/B_{r+1}$ is
$$A_{r+1}p^n(p^{nr+1} + \cdots + 1)/(B_{r+1}p^{nr+1}(p^n + \cdots + 1))$$

$$1 \times B_{s+1}p^n(p^{ns+1} + \cdots + 1)/(A_{s+1}p^{ns+1}(p^n + \cdots + 1)) \times \ldots \times B_{t}p^{n_{t-1}}(p^n + \cdots + 1)/\ldots \times B_{r}p^{n_{r-1}}(p^n + \cdots + 1)/A_{r+1}p^n(p^{nr+1} + \cdots + 1)/(B_{r+1}p^{nr+1}(p^n + \cdots + 1))$$

$$\times A_{x+1}p^{nx+1}(p^n + \cdots + 1)/(B_{x+1}p^{nx+1}(p^n + \cdots + 1)) = 1$$

$B_{s+1}B_{s+2} \ldots B_{t}A_{r+1}A_{r+2} \ldots A_{x}p^{n_{x-1}}(p^n + \cdots + 1)$

$= A_{s+1}A_{s+2} \ldots A_{r}B_{r+1}B_{r+2} \ldots B_{x}(p^n + \cdots + 1) \ldots (B)$

When $n_x < n$, it becomes contradiction since the right side of above expression does not include factor $p$.

When $n_x = n$,
$$B_{s+1}B_{s+2} \ldots B_{t}A_{r+1}A_{r+2} \ldots A_{x} = A_{s+1}A_{s+2} \ldots A_{r}B_{r+1}B_{r+2} \ldots B_{x} \ldots (C)$$
Let \( e_r, f_r \) be odd integers and \( g_r \) be a rational number,

\[
\begin{align*}
e_r &= \prod_{k=1}^{r} (p_k^{q_k} + \cdots + 1) \\
f_r &= \prod_{k=1}^{r} p_k^{q_k} \\
g_r &= e_r / f_r
\end{align*}
\]

holds.

\[ g_{r+1} = e_{r+1} / f_{r+1} = e_r / f_r \times (p_{r+1}^{q_{r+1}} + \cdots + 1) / p_{r+1}^{q_{r+1}} > e_r / f_r = g_r \]

Let \( q_1' \) be even integer and \( q_1' > q_1 \) holds. Let \( g_r \) be \( g_r' \) when \( q_1 \) becomes \( q_1' \).

\[ g_r' = (p_1^{q_1}(p_1^{q_1'} + \cdots + 1) / p_1^{q_1'}(p_1^{q_1} + \cdots + 1))g_r > g_r \]

is established.

Here, it is assumed that \( q_k \) becomes \( q_k - h_k \) by making \( q_k \) smaller than before for \( g_r, h_k \) is an even non-negative integer. Then it is assumed that \( r \) becomes \( s(s > r) \), \( g_s = g_r \) and \( g_s \) is not changed.

\[
\begin{align*}
g_{s/r} &= p_1^{q_1} \times \cdots \times p_r^{q_r}(p_1^{q_1 - h_1} + \cdots + 1) \cdots (p_r^{q_r - h_r} + \cdots + 1) / (p_1^{q_1} + \cdots + 1) \times \cdots \times (p_r^{q_r} + \cdots + 1) = 1 \\
p_1^{h_1} \times \cdots \times p_r^{h_r}(p_1^{q_1 - h_1} + \cdots + 1) \cdots (p_r^{q_r - h_r} + \cdots + 1) / ((p_1^{q_1} + \cdots + 1) \cdots (p_r^{q_r} + \cdots + 1)) \\
&\times p_{r+1}^{q_{r+1}} \times \cdots \times p_s^{q_s} = 1 \\
p_{r+1}^{q_{r+1}} \times \cdots \times p_s^{q_s} \times p_1^{h_1} \times \cdots \times p_r^{h_r}(p_1^{q_1 - h_1} + \cdots + 1) \cdots (p_r^{q_r - h_r} + \cdots + 1) \\
&= (p_1^{q_1} + \cdots + 1) \cdots (p_r^{q_r} + \cdots + 1) \\
&= (p_1^{q_1} + \cdots + 1) \cdots (p_r^{q_r} + \cdots + 1) \\
&= a = (p_1^{q_1} + \cdots + 1) \cdots (p_r^{q_r} + \cdots + 1) = cp^n \text{ holds and from the expression } 7, c \text{ must be a product of primes from } p_1 \text{ to } p_r. \text{ Thereby, the above equation does not hold since it is inappropriate when there is even one prime number other than } p_1 \text{ to } p_r. \text{ When changing the value of } p_k, \text{ it is equivalent to dividing by } p_k^{q_k} \text{ and then multiplying by new } p_k^{q_k}, \text{ so it is sufficient to consider only the changes of } q_k \text{ and } r. \text{ From above, since } g_r \text{ does not chord the original value when } q_k \text{ or } r \text{ is increased or decreased, it takes unique values for the variables } p_k, q_k, r. \text{ From above proof,}
\]

\[
\begin{align*}
g_r &= A_1A_2 \cdots A_s / B_1B_2 \cdots B_x \times A_{r+1}A_{r+2} \cdots A_x / B_{r+1}B_{r+2} \cdots B_x
\end{align*}
\]

g_r must be represented uniquely, and the expression (C) does not satisfied. When dividing by the prime number in the expression of \( p \), a contradiction arises since the prime number not included in \( b \) is in the expression of \( p \). Therefore, when \( p \) holds \( p \equiv 1 \pmod{4} \) and \( p \geq 5 \), the number of the solution \((a, b, p, n)\) satisfying \( R = 1 \) is at most one.
Define the operation [multiplication] and the operation [division] as follows.
Assuming that \( p \) in the equation of \( R \) is replaced by \( p' \) by multiplying \( A_i/B_i \), define operation [multiplication] to \( R \) as follows.

\[
p' = 2 \prod_{k=1}^{r} p_k^{d_k} \times p_i^{d_i} - 1
\]

\[0 \leq d_i \leq d_i\]

Here, let \( i \) be \( i > r \). Suppose operation [division] is division by \( A_i/B_i \) for \( R \), and if \( p_j \) is included in \( p \) in the expression \( R \), \( p_j \) is deleted as \( d_j = 0 \). Here, assuming that \( j \) satisfies \( 1 \leq j \leq r \).

In the proof of the expression (B), it is assumed that \( p \) changes on the way, and finally \( p \) becomes \( p_x \).

\[A_1 \ldots A_r = cp^n\]
\[2B_1 \ldots B_r = c(p^n + \cdots + 1)\]
\[A_1 \ldots A_x = c'p_x^n\]
\[2B_1 \ldots B_x = c'(p_x^n + \cdots + 1)\]

It is assumed that the above expressions are satisfied.

\[B_{s+1}B_{s+2} \ldots B_tA_{r+1}A_{r+2} \ldots A_xp^n(p_x^n + \cdots + 1)\]
\[B_{s+1}B_{s+2} \ldots B_tA_{r}A_{r+1}A_{r+2} \ldots A_xp^n(p_x^n + \cdots + 1)\]
\[B_{s+1}B_{s+2} \ldots B_tC_p^n(p_x^n + \cdots + 1)\]
\[B_{s+1}B_{s+2} \ldots B_tC_p^n(p_x^n + \cdots + 1)\]

\[c(p^n + \cdots + 1)/2 \times B_{s+1}B_{s+2} \ldots B_t p^n = c p^n A_{s+1}A_{s+2} \ldots A_r/2 \times (p^n + \cdots + 1)\]
\[B_{s+1}B_{s+2} \ldots B_t = A_{s+1}A_{s+2} \ldots A_r\]

is established. It becomes contradiction since \( A_k > B_k \) holds when the operation [division] is performed.
Since \((a, b, p, n) = (1, 1, 1, 1)\) is inappropriate solution and the expression (C) becomes contradiction, there is one solution when \(n_k = n = 1\). Therefore, there are no odd perfect numbers when \(n = 1\).

We consider in the case of \(n \geq 5\) as follows. Consider a tree whose vertex is \((a, b, p, n) = (1, 1, 1, 1)\), and it becomes a child node when the operation \([\text{multiplication}]\) is performed. For example, consider a child node connected to a vertex as follows.

\[
(a, b, p, n) = (13, 9, 5, 5) \text{ as } p_1 = 3, \ q_1 = 2 \text{ and } d_1 = 1
\]

\[
(a, b, p, n) = (13, 9, 17, 9) \text{ as } p_1 = 3, \ q_1 = 2 \text{ and } d_1 = 2
\]

\[
(a, b, p, n) = (57, 49, 97, 13) \text{ as } p_1 = 7, \ q_1 = 2 \text{ and } d_1 = 2
\]

The following lemma holds as a corollary of Zsigmondy's theorem. 
[lemma Z]
For odd prime \(p\) and odd \(n \geq 5\), where \(p \equiv 1, n \equiv 1 \pmod{4}\), \(p^{n+1} - 1\) has at least one prime factor different from any prime factor of \(p^2 - 1\).

By using this lemma Z, the following theorem can be proved.
[theorem]
For odd prime \(p\) and odd \(n \geq 5\), where \(p \equiv 1, n \equiv 1 \pmod{4}\), \(p^{n-1} + p^{n-3} + \cdots + p^2 + 1\) has a prime factor different from at least one prime factor of \((p + 1)/2\).

[proof]
From lemma Z, \(p^{n+1} - 1\) has at least one prime factor different from any prime factor of \(p^2 - 1\). Let this be \(q\).

\[p^{n+1} - 1 = (p^{n-1} + p^{n-3} + \cdots + p^2 + 1) \times (p^2 - 1)\] and since \(q\) is not a prime factor of \(p^2 - 1\), \(p^{n-1} + p^{n-3} + \cdots + p^2 + 1\) always has \(q\) as a prime factor. Since \(p^2 - 1\) is a multiple of \((p + 1)/2\), this \(q\) is different from any prime factor of \((p + 1)/2\). □

From the above, for odd prime number \(p\) and odd number \(n \geq 5\) where \(p \equiv 1, n \equiv 1 \pmod{4}\) \(p^{n-1} + p^{n-3} + \cdots + p^2 + 1\) can not be the product of only \((p + 1)/2\) prime factors.

We quoted the above lemma Z, theorem and proof from as below.
Proof for the existence of an odd complete number 3
https://rio2016.5ch.net/test/read.cgi/math/1544361065/498
From above theorem, when \( n \geq 5 \), if \( b \) is only a prime number of \((p + 1)/2\), it does not become an odd perfect number. ...(D)

It is assumed that a set of nodes is branched when \( p \) is changed by an operation [multiplication] in nodes in two or more layers. The order of the operation [multiplication] is such that the prime numbers changing the value of \( p \) come before the prime numbers not changing \( p \). Here, when there is a solution in a certain \( p \), if there is a solution even in the other values \( p' \), since there are no solutions in \( r = 1 \) and by proposition (D), the operation [division] must be performed to return to the bifurcation. At this time from above proof, it becomes contradiction. Thereby \( p \) must be unique. Therefore since for \( p \) satisfying \( p \geq 5 \) there is at most one solution with \( R = 1 \), the number of odd perfect number is one at most where \( n \geq 5 \).
4. Complement

From the equation (5),
\[ 2bp^n(p - 1) = a(p^{n+1} - 1) \]
\[ 2 = a(p^{n+1} - 1)/(bp^n(p - 1)) \]
\[ 2 = (p_1 q_1^{q_1 + 1} - 1)(p_2 q_2^{q_2 + 1} - 1) \ldots (p_r q_r^{q_r + 1} - 1)(p^{n+1} - 1) \]
\[ / (p_1 q_1 p_2 q_2 \ldots p_r q_r p^n(p_1 - 1)(p_2 - 1) \ldots (p_r - 1)(p - 1)) \]
\[ 2(p_1 q_1^{q_1} - 1)(p_2 q_2^{q_2} - 1) \ldots (p_r q_r^{q_r} - 1)(p^{n+1} - p^n) \]
\[ = (p_1 q_1^{q_1 + 1} - 1)(p_2 q_2^{q_2 + 1} - 1) \ldots (p_r q_r^{q_r + 1} - 1)(p^{n+1} - 1) \]

We consider when \( r = 2 \).
\[(p_1 q_1^{q_1 + 1} - 1)(p_2 q_2^{q_2 + 1} - 1)(p^{n+1} - 1) = 2(p_1 q_1^{q_1} - 1)(p_2 q_2^{q_2} - 1)(p^{n+1} - p^n) \]
Let \( s, t, u \) be integers,
\[ s = p_1 q_1^{q_1 + 1} - 1 \]
\[ t = p_2 q_2^{q_2 + 1} - 1 \]
\[ u = p^{n+1} - 1 \]
are.
\[ stu = 2(p_1 q_1^{q_1 + 1} - 1 - (p_1 q_1 - 1))(p_2 q_2^{q_2 + 1} - 1 - (p_2 q_2 - 1))(p^{n+1} - 1 - (p^n - 1)) \]
\[ stu = 2(s - (s + 1)/p_1 + 1)(t - (t + 1)/p_2 + 1)(u - (u + 1)/p + 1) \]
\[ pp_1 p_2 stu = 2((s + 1)p_1 - (s + 1))(t + 1)p_2 + (t + 1)((u + 1)p + (u + 1)) \]
\[ pp_1 p_2 stu = 2(s + 1)(p_1 - 1)(t + 1)(p_2 - 1)(u + 1)(p - 1) \]
\[ stu/((s + 1)(t + 1)(u + 1)) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) \]

Since \( stu/((s + 1)(t + 1)(u + 1)) \) is a monotonically increasing function for variables \( s, t \) and \( u \), if
\[ s \geq 3^{2^{q_1 + 1} - 1} = 26, p_1 = 3, q_1 = 2 \]
\[ t \geq 7^{2^{q_2 + 1} - 1} = 342, p_2 = 7, q_2 = 2 \]
\[ u \geq 5^{n+2} - 1 = 24, p = 5, n = 1 \]
holds,
\[ stu/((s + 1)(t + 1)(u + 1)) \geq 26 \times 342 \times 24/(27 \times 343 \times 25) = 7904/8575 \]
\[ 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35 \]
Since \( stu/((s + 1)(t + 1)(u + 1)) \) is limited to 1 when \( s, t \) and \( u \) are infinite, \( stu/((s + 1)(t + 1)(u + 1)) < 1 \)

If \( f(p_1,p_2,p) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p) \) holds, it is sufficient to consider a combination where \( f(p_1,p_2,p) < 1 \).

\[
\begin{align*}
f(3,7,5) &= 2 \times 2 \times 6 \times 4 / (3 \times 7 \times 5) = 32/35 \\ f(3,11,5) &= 2 \times 2 \times 10 \times 4 / (3 \times 11 \times 5) = 32/33 \\ f(3,13,5) &= 2 \times 2 \times 12 \times 4 / (3 \times 13 \times 5) = 64/65 \\ f(3,17,5) &= 2 \times 2 \times 16 \times 4 / (3 \times 17 \times 5) = 256/255 \\ f(3,7,13) &= 2 \times 2 \times 6 \times 12 / (3 \times 7 \times 13) = 96/91 \\ f(3,5,17) &= 2 \times 2 \times 4 \times 16 / (3 \times 5 \times 17) = 256/255 
\end{align*}
\]

From the above, when \( r = 2 \), a combination \((p_1,p_2,p) = (3,7,5),(3,11,5),(3,13,5)\) can be considered.

Let \( q_k \) be 2 and \( n = 1 \), if \( g(p_1,p_2,p) = (p_1^3 - 1)(p_2^3 - 1)(p^2 - 1)/(p_1^3p_2^3p^2) \),

\[
\begin{align*}
g(3,7,5) &= 26 \times 342 \times 24 / (3^37^35^2) = 7904/8575 > 32/35 \\ g(3,11,5) &= 26 \times 1330 \times 24 / (3^311^35^2) = 55328/59895 \\ g(3,13,5) &= 26 \times 2196 \times 24 / (3^313^35^2) = 3904/4225 
\end{align*}
\]

Since the function \( g \) is the minimum in the case of \( q_k = 2 \) and \( n = 1 \), there is no solution \( q_k \) and \( n \) when \( g > f \), so the case of \((p_1,p_2,p) = (3,7,5)\) becomes unsuitable.

\[
stu/((s + 1)(t + 1)(u + 1)) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p) \\
(p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1)(p^{n+1} - 1)/(p_1^{q_1+1}p_2^{q_2+1}p^{n+1}) \\
= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p)
\]

If \( F(p_1,p_2,p) = (p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p) \),

\( F(p_1^{q_1+1},p_2^{q_2+1},p^{n+1}) = 2F(p_1,p_2,p) \)
5. Acknowledgement

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6. References

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