Proof that there are no odd perfect numbers

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1. Abstract

For $y$ to be a perfect number, if one of the prime factors is $p$, the exponent of $p$ is an integer $n(n \geq 1)$, the prime factors other than $p$ are $p_1, p_2, p_3, \cdots p_r$ and the even exponent of $p_k$ is $q_k$,

$$y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k})/(2p^n) = \prod_{k=1}^{r} p_k^{q_k}$$

must be satisfied. Let $m$ be non negative integer and $q$ be positive integer,

$$n = 4m + 1$$
$$p = 4q + 1$$

Letting $b$ and $c$ be odd integers, satisfying following expressions,

$$b = \prod_{k=1}^{r} p_k^{q_k}$$
$$c = \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k})/p^n$$

$$2b = c(p^n + \cdots + 1)$$

is established. This is a known content. By the consideration of this research paper, since it turned out that the solution $(a, b, p, n)$ that satisfies this equation is at most one and that one is an inappropriate solution, we have obtained the conclusion that there are no odd perfect numbers.

2. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself, and the smallest perfect number is

$$1 + 2 + 3 = 6$$

It is 6. Whether an odd perfect number exists or not is currently an unsolved problem.
3. Proof

An odd perfect number is y, one of them is an odd prime number p, an exponent of p is an integer n \((n \geq 1)\). Let \(p_1, p_2, p_3, \ldots, p_r\) be the odd prime numbers of factors other than p, \(q_k\) the index of \(p_k\), and variable a be the sum of product combinations other than prime p.

\[
a = \prod_{k=1}^{r}(1 + p_k + p_k^2 + \cdots + p_k^{q_k}) \quad \text{①}
\]

The number of terms \(N\) of variable a is

\[
N = \prod_{k=1}^{r}(q_k + 1) \quad \text{②}
\]

When y is a perfect number,

\[
y = a(1 + p + p^2 + \cdots + p^n) - y \quad (n > 0)
\]

is established.

\[
a \sum_{k=0}^{n} p^k / 2 = y
\]

\[
a \sum_{k=0}^{n} p^k / (2p^n) = y/p^n \quad \text{③}
\]

3.1. If \(q_k\) has at least one odd integer

Letting the number of terms where \(q_k\) is an odd integer be a positive integer \(u\), because \(y/p^n = \prod_{k=1}^{r} p_k^{q_k}\) is an odd integer, the denominator on the left side of expression ③ has a prime factor 2, from expression ② variable a has more than \(u\) prime factor 2 and variable a is an even integer. Therefore \(\sum_{k=0}^{n} p^k\) must be an odd integer, \(n\) is an even integer and \(u\) is 1.

3.2. When all \(q_k\) are even integers

\(y/p^n\) is an odd integer, the denominator on the left side of expression ③ is an even integer, and since \(N\) is an odd integer when \(q_k\) are all even integers, variable a is an odd integer. Therefore \(\sum_{k=0}^{n} p^k\) is necessary to include one prime factor 2, \(\sum_{k=0}^{n} p^k \equiv 0 \pmod{2}\) is established, and \(n\) must be an odd integer.

From 3.1, 3.2, in order to have an odd perfect number, only one exponent of the prime factor of y must be an odd integer and variable a must be an odd integer. We consider the case of 3.2 below.
In order for \( y \) to be a perfect number, the following expression must be established:

\[
y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k}) / (2p^n) = \prod_{k=1}^{r} p_k^{q_k}
\]

However, \( q_1, q_2, \ldots, q_r \) are all even integers.

Here, let \( b \) be an integer

\[
b = \prod_{k=1}^{r} p_k^{q_k} \quad \cdots \quad (4)
\]

A following expression is established.

\[
y/p^n = a(1 + p + p^2 + \cdots + p^n)/(2p^n) = b
\]

\[
a(p^{n+1} - 1)/(2(p - 1)p^n) = b
\]

\[
(a - 2b)p^{n+1} + 2bp^n - a = 0 \quad \cdots \quad (5)
\]

Because it is an \( n + 1 \) order equation of \( p \), the solution of the odd prime \( p \) is \( n + 1 \) at most.

\[
(ap - 2bp + 2b)p^n = a
\]

Since \( ap - 2bp + 2b \) is an odd integer, \( a/p^n \) is an odd integer, which is \( c \).

\[
ap - 2bp + 2b = c \quad (c > 0) \quad \cdots \quad (6)
\]

\[
(2b - a)p = 2b - c
\]

Since variable \( a \) is an odd integer, \( 2b - a \) is an odd integer and \( 2b - a \neq 0 \)

\[p = (2b - c)/(2b - a)\]
Since \( n \geq 1 \)
\[ a - c = cp^n - c \geq cp - c > 0 \]
\[ a > c \]
is.

From equation ⑥
\[ 2b(p - 1) - (ap - c) = 0 \]
\[ 2b - c(p^{n+1} - 1)/(p - 1) = 0 \]
\( (p^n + \cdots + 1)/2 \) is an odd integer, \( n = 4m + 1 \) is required with \( m \) as an integer.
\[ 2b(p - 1) = c(p^{n+1} - 1) \]
\[ 2b = c(p^n + \cdots + 1) \]
\[ 2b = c(p + 1)(p^{n-1} + p^{n-3} + \cdots + 1) \ldots ⑦ \]
b is an odd integer when \( p + 1 \) is not a multiple of 4. It is necessary that \( p - 1 \) be a multiple of 4. A positive integer is taken as \( q \).
\[ p = 4q + 1 \]
is established.

When \( p > 1 \)
\[ p^n - 1 < p^n \]
\[ (p^n - 1)/(p - 1) < p^n/(p - 1) \]
\[ p^{n-1} + \cdots + 1 < p^n/(p - 1) \ldots ⑧ \]

Since \( p \) is an odd prime number satisfying \( p = 4q + 1 \) and \( p \geq 5 \)
\[ p^{n-1} + \cdots + 1 < p^n/4 \]
\[ 2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1) \]
\[ 2b - a < cp^n/4 = a/4 \]
\[ 2b < 5a/4 \]
\[ a > 8b/5 \ldots ⑨ \]
Let $a_k$ and $b_k$ be integers and if
$a_k = 1 + p_k + p_k^2 + \cdots + p_k^{q_k}$, $b_k = p_k^{q_k}$,
$a_k - b_k < b_k/(p_k - 1)$
a_k < b_k p_k/(p_k - 1)

\[
a = \prod_{k=1}^r a_k < \prod_{k=1}^r b_k p_k/(p_k - 1) = b \prod_{k=1}^r p_k/(p_k - 1)
\]
a/b < \prod_{k=1}^r p_k/(p_k - 1)

When $r = 1$, since $a/b < 3/2$ is established, it becomes inappropriate contrary to inequality ⑨.

From expression ⑦,
\[b = c(p + 1)/2 \times (p^{n-1} + p^{n-3} + \cdots + 1)\]
holds. Since $(p + 1)/2$ is the product of only prime numbers of $b$, let $d_k$ be the index,
\[(p + 1)/2 = \prod_{k=1}^r p_k^{d_k}\]
\[p = 2 \prod_{k=1}^r p_k^{d_k} - 1\]

From $a = cp^n$ and expression ⑦,
\[2bp^n = a(p^n + \cdots + 1)\]
\[a(p^n + \cdots + 1)/(2bp^n) = 1 \quad \text{...(A)}\]
When $r = 1$,
\[a = (p_1^{q_1+1} - 1)/(p_1 - 1)\]
\[b = p_1^{q_1}\]
Equation (A) does not hold since there is no odd perfect number when $r = 1$. 

Let $R$ be a rational number,
\[ R = a(p^n + \cdots + 1)/(2bp^n) \]
Let $b'$ be a rational number and let $A$ and $B$ to be an integer,
\[ b' = (p_k^{q_k+1} - 1)/(p_k^{q_k}(p_k - 1)) > 1 \]
\[ A_k = (p_k^{q_k+1} - 1)/(p_k - 1) \]
\[ B_k = p_k^{q_k} \]

Multiplying $R$ by $b'$, there are both cases that $p_k$ increases $p$ or does not change. When multiplied by $b'$, the rate of change of $R$ is $A_{r+1}p^n(p^{r+1} + \cdots + 1)/(B_{r+1}p^n(p^n + \cdots + 1))$, if $p$ after variation is $p'$. If the rate of change of $R$ is 1,
\[ A_{r+1}p^n(p^n + \cdots + 1)/(B_{r+1}p^n(p^n + \cdots + 1)) = 1 \]
\[ A_{r+1}p^n(p^n + \cdots + 1) = B_{r+1}p^n(p^n + \cdots + 1) \]
This expression does not hold since the right side is not a multiple of $p$ when $p' > p$, and $A_{r+1} > B_{r+1}$ holds when $p' = p$. Due to this operation, $R$ may be larger or smaller than the original value since the rate of change of $R$ does not become 1.

Assuming that $R = 1$ in some $r$, letting $x$ be an integer and by multiplying fractions $b' = A_{r+1}/B_{r+1}$, $b'' = A_{r+2}/B_{r+2}$, $\cdots b'' = A_x/B_x$ to $R$. Furthermore, assuming that $A_{x+1}A_{x+2} \cdots A_r$ is not a multiple of $p$, $R$ is divided by $A_{x+1}/B_{x+1}$, $A_{x+2}/B_{x+2}$, $\cdots A_r/B_r$ and it is assumed that finally $R = 1$. At this time, assuming that $n$ changes, the change rate of $R$ by this operation when multiplying by $A_{r+1}/B_{r+1}$ is
\[ A_{r+1}p^n(p^{nr+1} + \cdots + 1)/(B_{r+1}p^{nr+1}(p^n + \cdots + 1)) \]
\[ 1 \times B_{s+1}p^n(p^{ns+1} + \cdots + 1)/(A_{s+1}p^{ns+1}(p^n + \cdots + 1)) \times \cdots \times B_{t}p^{nr-1}(p^{nr + 1}/(A_{r}p^{nr}(p^{nr+1} + \cdots + 1)) \times A_{r+1}p^n(p^{nr+1} + \cdots + 1)/(B_{r+1}p^{nr+1}(p^{nr + 1}) \times A_{r+2}p^{nr+2}(p^{nr+2} + \cdots + 1)/(B_{r+2}p^{nr+2}(p^{nr+2} + \cdots + 1)) \times \cdots \times A_{x}p^{nx}(p^{nx} + \cdots + 1)/(B_{x}p^{nx}(p^{nx} + \cdots + 1)) = 1 \]
\[ B_{s+1}B_{s+2} \cdots B_{r}A_{r+1}A_{r+2} \cdots A_{x}p^{n-x}(p^{nx} + \cdots + 1) \]
\[ = A_{s+1}A_{s+2} \cdots A_{r}B_{r+1}B_{r+2} \cdots B_{x}(p^n + \cdots + 1) \] (B)
When $n_x < n$, it becomes contradiction since the right side of above expression does not include factor $p$.
When $n_x = n$,
\[ B_{s+1}B_{s+2} \cdots B_{r}A_{r+1}A_{r+2} \cdots A_{x} = A_{s+1}A_{s+2} \cdots A_{r}B_{r+1}B_{r+2} \cdots B_{x} \] (C)
Let $e_r$, $f_r$ be odd integers and $g_r$ be a rational number,
\[ e_r = \prod_{k=1}^{r} (p_k^{q_k} + \cdots + 1) \]
\[ f_r = \prod_{k=1}^{r} p_k^{q_k} \]
\[ g_r = e_r / f_r \]
holds.

Let $q_1'$ be even integer and $q_1 > q_1$ holds. Let $g_r$ be $g_r'$ when $q_1$ becomes $q_1'$,
\[ g_r' = (p_1 q_1^r (p_1 q_1' + \cdots + 1)/p_1 q_1' (p_1 q_1 + \cdots + 1)) g_r > g_r \]
is established.

Here, it is assumed that $q_k$ becomes $q_k - h_k$ by making $q_k$ smaller than before for $g_r$, $h_k$ is an even non-negative integer. Then it is assume that $r$ becomes $s(s > r)$,
\[ g_s = g_r \text{ and } g_s \text{ is not changed.} \]
\[ g_s/g_r = p_1 q_1 \times \cdots \times p_r q_r (p_1 q_1^{-h_1} + \cdots + 1) \cdots (p_r q_r^{-h_r} + \cdots + 1)/(p_1 q_1^{-h_1} \times \cdots \times p_r q_r^{-h_r}) = 1 \]
\[ p_1^{-h_1} \times \cdots \times p_r^{-h_r} (p_1 q_1^{-h_1} + \cdots + 1) \cdots (p_r q_r^{-h_r} + \cdots + 1) = \]
\[ \cdot p_{r+1} q_{r+1} \times \cdots \times p_s q_s = 1 \]
\[ p_{r+1} q_{r+1} \times \cdots \times p_s q_s \times p_1^{-h_1} \times \cdots \times p_r^{-h_r} (p_1 q_1^{-h_1} + \cdots + 1) \cdots (p_r q_r^{-h_r} + \cdots + 1) = \]
\[ = (p_1 q_1 + \cdots + 1) \cdots (p_r q_r + \cdots + 1) \]
\[ a = (p_1 q_1 + \cdots + 1) \cdots (p_r q_r + \cdots + 1) = cp^n \text{ holds and from expression } 7, c \text{ must be a product of primes from } p_1 \text{ to } p_r. \]

Therefore, the above equation does not hold since it is inappropriate when there is even one prime number other than $p_1$ to $p_r$. When changing the value of $p_k$, it is equivalent to dividing by $p_k q_k$ and then multiplying by new $p_k q_k$, so it is sufficient to consider only the changes of $q_k$ and $r$. From above, since $g_r$ does not chord the original value when $q_k$ or $r$ is increased or decreased, it takes unique values for the variables $p_k$, $q_k$, $r$.

From above proof,
\[ g_r = A_1 A_2 \cdots A_{r-1}/B_1 B_2 \cdots B_{r-1} \times A_{s+1} A_{s+2} \cdots A_r/B_{s+1} B_{s+2} \cdots B_r \]
is represented uniquely, and expression (C) does not satisfied. When dividing by the prime number in the expression of $p$, a contradiction arises since the prime number not included in $b$ is in the expression of $p$. Therefore, when $p$ holds $p \equiv 1 \pmod{4}$ and $p \geq 5$, the number of the solution $(a,b,p,n)$ satisfying $R = 1$ is at most one.
In the proof of expression (B), it is assumed that p changes on the way, and finally p becomes \( p_x \).

\[
A_1 \ldots A_r = cp^n
\]

\[
2B_1 \ldots B_r = c(p^n + \cdots + 1)
\]

\[
A_1 \ldots A_x = c'p_x^n
\]

\[
2B_1 \ldots B_x = c'(p_x^n + \cdots + 1)
\]

It is assumed that the above expressions are satisfied.

\[
B_{s+1}B_{s+2} \ldots B_r A_{r+1}A_{r+2} \ldots A_x p^n (p_x^{n_x} + \cdots + 1)
\]

\[
= A_{s+1}A_{s+2} \ldots A_r B_{r+1}B_{r+2} \ldots B_x p_x^{n_x} (p^n + \cdots + 1)
\]

\[
= A_1 \ldots A_r A_{s+1}A_{s+2} \ldots A_x B_{r+1}B_{r+2} \ldots B_x p_x^{n_x} (p^n + \cdots + 1)
\]

\[
B_{s+1}B_{s+2} \ldots B_r c'p_x^{n_x} p^n (p_x^{n_x} + \cdots + 1)
\]

\[
= A_1 \ldots A_r A_{s+1}A_{s+2} \ldots A_x B_{r+1}B_{r+2} \ldots B_x p_x^{n_x} (p^n + \cdots + 1)
\]

\[
B_{s+1}B_{s+2} \ldots B_r c'p^n (p_x^{n_x} + \cdots + 1) = A_1 \ldots A_r A_{s+1}A_{s+2} \ldots A_x B_{r+1}B_{r+2} \ldots B_x (p^n + \cdots + 1)
\]

\[
B_1 \ldots B_r B_{s+1}B_{s+2} \ldots B_r c'p^n (p_x^{n_x} + \cdots + 1)
\]

\[
= A_1 \ldots A_r A_{s+1}A_{s+2} \ldots A_x B_1 \ldots B_r B_{r+1}B_{r+2} \ldots B_x (p^n + \cdots + 1)
\]

\[
B_1 \ldots B_r B_{s+1}B_{s+2} \ldots B_r c'p^n (p_x^{n_x} + \cdots + 1)
\]

\[
= A_1 \ldots A_r A_{s+1}A_{s+2} \ldots A_x c' (p_x^{n_x} + \cdots + 1) / 2 \times (p^n + \cdots + 1)
\]

\[
B_1 \ldots B_r B_{s+1}B_{s+2} \ldots B_r p^n = A_1 \ldots A_r A_{s+1}A_{s+2} \ldots A_r / 2 \times (p^n + \cdots + 1)
\]

\[
c(p^n + \cdots + 1) / 2 \times B_{s+1}B_{s+2} \ldots B_r p^n = cp^n A_{s+1}A_{s+2} \ldots A_r / 2 \times (p^n + \cdots + 1)
\]

\[
B_{s+1}B_{s+2} \ldots B_r = A_{s+1}A_{s+2} \ldots A_r \ldots (D)
\]

is established. It becomes contradiction since \( A_k > B_k \) holds. Thus, the number of solutions \((a, b, p, n)\) for which \( R = 1 \) does not depend on the values of \( p \) and \( n \) is one at most. However, since \((a, b, p, n) = (1, 1, 1, 1)\) becomes a solution, there is not any solution other than this combination. When the division is not performed, the above expression holds. From expression (B),

\[
A_{r+1}A_{r+2} \ldots A_x p^{n-n_x}(p^{n_x} + \cdots + 1) = B_{r+1}B_{r+2} \ldots B_x (p^n + \cdots + 1)
\]

When \( n_x < n \), it becomes contradiction for \( p \geq 5 \) since the right side of above expression does not include factor \( p \).

When \( n_x = n \),

\[
A_{r+1}A_{r+2} \ldots A_x = B_{r+1}B_{r+2} \ldots B_x
\]

It becomes contradiction. Thereby, when expression (D) is satisfied, it becomes inappropriate since \( p \) must be \( p = p_x = 1 \). Therefore, there are no odd perfect numbers.
4. Complement

From equation \( 5 \),
\[
2bp^n(p - 1) = a(p^{n+1} - 1)
\]
2 = \((p_1 q_1 + 1)(p_2 q_2 + 1) \ldots (p_r q_r + 1)(p^{n+1} - 1)\)
\[
/p_1 q_1 p_2 q_2 \ldots p_r q_r p^n(p_1 - 1) \ldots (p_r - 1)(p - 1)\)
2 \((p_1 q_1 + 1)(p_2 q_2 + 1) \ldots (p_r q_r + 1)(p^{n+1} - 1)\)
\[
= (p_1 q_1 + 1)(p_2 q_2 + 1) \ldots (p_r q_r + 1)(p^{n+1} - 1)\)

We consider when \( r = 2 \).
\[(p_1 q_1 + 1)(p_2 q_2 + 1)(p^{n+1} - 1) = 2(p_1 q_1 + 1)(p_2 q_2 + 1)(p^{n+1} - 1)\]

Let \( s, t, u \) be integers,
\[
s = p_1 q_1 + 1
\]
\[
t = p_2 q_2 + 1
\]
\[
u = p^{n+1} - 1
\]
are.
\[
stu = 2(p_1 q_1 + 1 - (p_1 q_1 - 1))(p_2 q_2 + 1 - (p_2 q_2 - 1))(p^{n+1} - 1 - (p^n - 1))
\]
\[
stu = 2(s - (s + 1)/p_1 + 1)(t - (t + 1)/p_2 + 1)(u - (u + 1)/p + 1)
\]
\[
pp_1 p_2 stu = 2((s + 1)p_1 - (s + 1))(t + 1)p_2 + (t + 1)((u + 1)p + (u + 1))
\]
\[
pp_1 p_2 stu = 2(s + 1)(p_1 - 1)(p_2 - 1)(u + 1)(p - 1)
\]
\[
stu/((s + 1)(t + 1)(u + 1)) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)
\]

Since \( stu/((s + 1)(t + 1)(u + 1)) \) is a monotonically increasing function for variables \( s, t \) and \( u \), if
\[
s \geq 2^{2+1} - 1 = 26, \ p_1 = 3, \ q_1 = 2
\]
\[
t \geq 7^{2+1} - 1 = 342, \ p_2 = 7, \ q_2 = 2
\]
\[
u \geq 5^2 - 1 = 24, \ p = 5, \ n = 1
\]
holds,
\[
stu/((s + 1)(t + 1)(u + 1)) \geq 26 \times 342 \times 24/(27 \times 343 \times 25) = 7904/8575
\]
\[
2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35
\]
Since \( stu/((s+1)(t+1)(u+1)) \) is limited to 1 when \( s, t \) and \( u \) are infinite, \\
\( stu/((s+1)(t+1)(u+1)) < 1 \)

If \( f(p_1, p_2, p) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) \) holds, it is sufficient to consider a combination where \( f(p_1, p_2, p) < 1 \).

\[
\begin{align*}
 f(3,7,5) &= 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35 \\
 f(3,11,5) &= 2 \times 2 \times 10 \times 4/(3 \times 11 \times 5) = 32/33 \\
 f(3,13,5) &= 2 \times 2 \times 12 \times 4/(3 \times 13 \times 5) = 64/65 \\
 f(3,17,5) &= 2 \times 2 \times 16 \times 4/(3 \times 17 \times 5) = 256/255 \\
 f(3,7,13) &= 2 \times 2 \times 6 \times 12/(3 \times 7 \times 13) = 96/91 \\
 f(3,5,17) &= 2 \times 2 \times 4 \times 16/(3 \times 5 \times 17) = 256/255
\end{align*}
\]

From the above, when \( r = 2 \), a combination \((p_1, p_2, p) = (3,7,5), (3,11,5), (3,13,5)\) can be considered.

Let \( q_k \) be 2 and \( n = 1 \), if \( g(p_1, p_2, p) = (p_1^3 - 1)(p_2^3 - 1)(p^n - 1)/(p_1^3 p_2^3 p^n) \),

\[
\begin{align*}
 g(3,7,5) &= 26 \times 342 \times 24/(3^3 7^3 5^2) = 7904/8575 > 32/35 \\
 g(3,11,5) &= 26 \times 1330 \times 24/(3^3 11^3 5^2) = 55328/59895 \\
 g(3,13,5) &= 26 \times 2196 \times 24/(3^3 13^3 5^2) = 3904/4225
\end{align*}
\]

Since the function \( g \) is the minimum in the case of \( q_k = 2 \) and \( n = 1 \), there is no solution \( q_k \) and \( n \) when \( g > f \), so the case of \((p_1, p_2, p) = (3,7,5)\) becomes unsuitable.

\[
\begin{align*}
stu/((s+1)(t+1)(u+1)) &= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) \\
(p_1 q_1 + 1)(p_2 q_2 + 1)(p^n + 1)/(p_1 q_1 + 1 p_2 q_2 + 1 p^n + 1) \\
&= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)
\end{align*}
\]

If \( F(p_1, p_2, p) = (p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p) \),

\[
F(p_1 q_1 + 1, p_2 q_2 + 1, p^n + 1) = 2F(p_1, p_2, p)
\]

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6. References
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