Proof that there are no odd perfect numbers

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1. Abstract

For y to be a perfect number, if one of the prime factors is p, the exponent of p is an integer $n(n \geqq 1)$, the prime factors other than p are p_1,p_2,p_3,$\cdots p_r$ and the even exponent of p_k is q_k,

$$y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} (1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k}) /(2p^n) = \prod_{k=1}^{r} {p_k}^{q_k}$$

must be satisfied. Let m and q be non-negative integers,

$$n = 4m + 1$$

$$p = 4q + 1$$

Letting b and c be odd integers, satisfying following expressions,

$$b = \prod_{k=1}^{r} {p_k}^{q_k}$$

$$c = \prod_{k=1}^{r} (1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k})/p^n$$

$$2b = c(p^n + \cdots + 1)$$

is established. This is a known content. By the consideration of this research paper, since it turns out that if the odd perfect number exists, the number of it is at most one and it becomes a solution at infinity, we have obtained the conclusion that there are no odd perfect numbers.

2. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself, and the smallest perfect number is

$$1 + 2 + 3 = 6$$

It is 6. Whether an odd perfect number exists or not is currently an unsolved problem.

3. Proof

An odd perfect number is y, one of them is an odd prime number p, an exponent of p is an integer n (n ≧ 1). Let $p_1, p_2, p_3, \cdots p_r$ be the odd prime numbers of factors other than p, q_k the index of p_k, and variable a be the sum of product combinations other than prime p.

$$a = \prod_{k=1}^{r} (1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k}) \quad ...①$$

The number of terms N of variable a is

$$N = \prod_{k=1}^{r} (q_k + 1) \quad ...②$$

When y is a perfect number,

$$y = a(1 + p + p^2 + \cdots + p^n) - y \ (n > 0)$$

is established.

$$a \sum_{k=0}^{n} p^k /2 = y$$

$$a \sum_{k=0}^{n} p^k /(2p^n) = y/p^n \quad ...③$$

3.1. If q_k has at least one odd integer

Letting the number of terms where q_k is an odd integer be a positive integer u, because $y/p^n = \prod_{k=1}^{r} {p_k}^{q_k}$ is an odd integer, the denominator on the left side of expression ③ has a prime factor 2, from expression ② variable a has more than u prime factor 2 and variable a is an even integer. Therefore $\sum_{k=0}^{n} p^k$ must be an odd integer, n is an even integer and u is 1.

3.2. When all q_k are even integers

y/p^n is an odd integer, the denominator on the left side of expression ③ is an even integer, and since N is and odd integer when q_k are all even integers, variable a is and odd integer. Therefore $\sum_{k=0}^{n} p^k$ is necessary to include one prime factor 2, $\sum_{k=0}^{n} p^k \equiv 0 \ (mod\ 2)$ is established, and n must be an odd integer.

From 3.1, 3.2, in order to have an odd perfect number, only one exponent of the prime factor of y must be an odd integer and variable a must be an odd integer. We consider the case of 3.2 below.

In order for y to be a perfect number, the following expression must be established.

$$y/p^n = (1 + p + p^2 + \cdots + p^n)\prod_{k=1}^{r}(1 + p_k + {p_k}^2 + \cdots + {p_k}^{q_k})/(2p^n) = \prod_{k=1}^{r}{p_k}^{q_k}$$

However, $q_1, q_2, \dots, q_r$ are all even integers.

Here, let b be an integer

$$b = \prod_{k=1}^{r}{p_k}^{q_k} \quad \dots ④$$

A following expression is established.

$y/p^n = a(1 + p + p^2 + \cdots + p^n)/(2p^n) = b$

$a(p^{n+1} - 1)/(2(p - 1)p^n) = b$

$(a - 2b)p^{n+1} + 2bp^n - a = 0$ ...⑤

Because it is an $n + 1$ order equation of p, the solution of the odd prime p is $n + 1$ at most.

$(ap - 2bp + 2b)p^n = a$

Since $ap - 2bp + 2b$ is an odd integer, a/p^n is an odd integer, which is c.

$ap - 2bp + 2b = c \ (c > 0)$ ...⑥

$(2b - a)p = 2b - c$

Since variable a is an odd integer, $2b - a$ is an odd integer and $2b - a \neq 0$

$p = (2b - c)/(2b - a)$

Since $n \geqq 1$

$a - c = cp^n - c \geqq cp - c > 0$

$a > c$

is.

From equation ⑥

$2b(p-1) - (ap - c) = 0$

$2b - c(p^{n+1} - 1)/(p-1) = 0$

$(p^n + \cdots + 1)/2$ is an odd integer, $n = 4m + 1$ is required with m as an integer.

$2b(p-1) = c(p^{n+1} - 1)$

$2b = c(p^n + \cdots + 1)$

$2b = c(p+1)(p^{n-1} + p^{n-3} + \cdots + 1)$ ...⑦

b is an odd integer when $p + 1$ is not a multiple of 4. It is necessary that $p - 1$ be a multiple of 4. A positive integer is taken as q.

$p = 4q + 1$

is established.

When $p > 1$

$p^n - 1 < p^n$

$(p^n - 1)/(p-1) < p^n/(p-1)$

$p^{n-1} + \cdots + 1 < p^n/(p-1)$ ...⑧

Since p is an odd prime number satisfying $p = 4q + 1$ and $p \geqq 5$

$p^{n-1} + \cdots + 1 < p^n/4$

$2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1)$

$2b - a < cp^n/4 = a/4$

$2b < 5a/4$

$a > 8b/5$ ...⑨

Let a_k and b_k be integers and if

$a_k = 1 + p_k + p_k^2 + \cdots + p_k^{q_k}$, $b_k = p_k^{q_k}$,

$a_k - b_k < b_k/(p_k - 1)$

$a_k < b_k p_k/(p_k - 1)$

$$a = \prod_{k=1}^{r} a_k < \prod_{k=1}^{r} b_k p_k/(p_k - 1) = b \prod_{k=1}^{r} p_k/(p_k - 1)$$

$$a/b < \prod_{k=1}^{r} p_k/(p_k - 1)$$

When $r = 1$, since $a/b < 3/2$ is established, it becomes inappropriate contrary to inequality ⑨.

From expression ⑦,

$b = c(p + 1)/2 \times (p^{n-1} + p^{n-3} + \cdots + 1)$

holds. Since $(p + 1)/2$ is the product of only prime numbers of b, let d_k be the index,

$$(p + 1)/2 = \prod_{k=1}^{r} {p_k}^{d_k}$$

$$p = 2 \prod_{k=1}^{r} {p_k}^{d_k} - 1$$

From $a = cp^n$ and expression ⑦,

$2bp^n = a(p^n + \cdots + 1)$

$a(p^n + \cdots + 1)/(2bp^n) = 1$ ... (A)

When $r = 1$,

$a = ({p_1}^{q_1+1} - 1)/(p_1 - 1)$

$b = {p_1}^{q_1}$

Equation (A) does not hold since there is no odd perfect number when $r = 1$.

Let R be a rational number,

$R = a(p^n + \cdots + 1)/(2bp^n)$

Let b' be a rational number and let A and B to be an integer,

$b' = (p_k{}^{q_k+1} - 1)/(p_k{}^{q_k}(p_k - 1)) > 1$

$A = (p_k{}^{q_k+1} - 1)/(p_k - 1)$

$B = p_k{}^{q_k}$

Multiplying R by b', there are both cases that p_k increases p or does not change. When multiplied by b', the rate of change of R is $Ap^n(p'^n + \cdots + 1)/(Bp'^n(p^n + \cdots + 1))$, if p after variation is p'. If the rate of change of R is 1,

$Ap^n(p'^n + \cdots + 1)/(Bp'^n(p^n + \cdots + 1)) = 1$

$Ap^n\left(p'^n + \cdots + 1\right) = Bp'^n(p^n + \cdots + 1)$

This expression does not hold, since the right side is not a multiple of p when $p' > p$, and $A > B$ holds when $p' = p$. Due to this operation, R may be larger or smaller than the original value, since the rate of change of R does not become 1.

From $R \neq 1$ and $a = cp^n$ for some r, also multiplying fractions $b' = A_1/B_1$, $b'' = A_2/B_2$, $\cdots b''^{\cdots\prime} = A_x/B_x$, if $R = 1$ holds finally,

, if $a(p^n + \cdots + 1)/(2bp^n) \times A_1p^n(p_1{}^n + \cdots + 1)/(B_1p_1{}^n(p^n + \cdots + 1)) \times A_2p_1{}^n(p_2{}^n + \cdots + 1)/(B_2p_2{}^n(p_1{}^n + \cdots + 1)) \ldots A_xp_{x-1}{}^n(p_x{}^n + \cdots + 1)/(B_xp_x{}^n(p_{x-1}{}^n + \cdots + 1)) = 1$

$a/(2b) \times A_1/B_1 \times A_2/B_2 \ldots A_x(p_x{}^n + \cdots + 1)/(B_xp_x{}^n) = 1$

$a(p_x{}^n + \cdots + 1)A_1A_2 \ldots A_x = 2bp_x{}^nB_1B_2 \ldots B_x$

$cp^n(p_x{}^n + \cdots + 1)A_1A_2 \ldots A_x = 2bp_x{}^nB_1B_2 \ldots B_x$

When $p_x > p$, it becomes inconsistent since the right side of this expression does not include p as a factor.

When $p_x = p$,

$cp^n(p^n + \cdots + 1)A_1A_2 \ldots A_x = c(p^n + \cdots + 1)p^n$

$A_1A_2 \ldots A_x = 1$

It becomes contradiction, since this expression is not established. Therefore, $a = cp^n$ holds at one point where $R = 1$.

Assuming that $R = 1$ in some r by multiplying fractions $b' = A_1/B_1$, $b'' = A_2/B_2$, $\cdots$ $b''^{\cdots\prime} = A_x/B_x$, if $R = 1$ holds,

$$1 \times A_1 p^n(p_1{}^n + \cdots + 1)/(B_1 p_1{}^n(p^n + \cdots + 1)) \times A_2 p_1{}^n(p_2{}^n + \cdots + 1)/(B_2 p_2{}^n(p_1{}^n + \cdots + 1)) \ldots A_x p_{x-1}{}^n(p_x{}^n + \cdots + 1)/(B_x p_x{}^n(p_{x-1}{}^n + \cdots + 1)) = 1$$

$$A_1 A_2 \ldots A_x p^n(p_x{}^n + \cdots + 1) = B_1 B_2 \ldots B_x p_x{}^n(p^n + \cdots + 1)$$

When $p_x > p$, it becomes inconsistent since the right side of this expression does not include p as a factor.

When $p_x = p$,

$$A_1 A_2 \ldots A_x = B_1 B_2 \ldots B_x$$

is established. It becomes contradiction. Therefore, when n is fixed, the number of values of r for which $R = 1$ is one or less.

Assuming that $R = 1$ in some r by multiplying fractions $b' = A_1/B_1$, $b'' = A_2/B_2$, $\cdots$ $b''^{\cdots\prime} = A_x/B_x$ and reciprocal of fraction previously multiplied ,if $R = 1$ holds. At this time, assuming that n also changes, the change rate when multiplying by A_1/B_1 is

$$A_1 p^n(p_{r+1}{}^{n_{r+1}} + \cdots + 1)/(B_1 p_{r+1}{}^{n_{r+1}}(p^n + \cdots + 1))$$

The rate of change when multiplying $p_{r+2}{}^{q_{r+2}}/(p_{r+2}{}^{q_{r+2}} + \cdots + 1)$ after this is

$$p_1{}^{q_1} p_{r+1}{}^{n_{r+1}}(p_{r+2}{}^{n_{r+2}} + \cdots + 1)/((p_1{}^{q_1} + \cdots + 1)p_{r+2}{}^{n_{r+2}}(p_{r+1}{}^{n_{r+1}} + \cdots + 1))$$

$$1 \times A_1 p^n(p_{r+1}{}^{n_{r+1}} + \cdots + 1)/(B_1 p_{r+1}{}^{n_{r+1}}(p^n + \cdots + 1)) \times p_1{}^{q_1} p_{r+1}{}^{n_{r+1}}(p_{r+2}{}^{n_{r+2}} + \cdots + 1)/((p_1{}^{q_1} + \cdots + 1)p_{r+2}{}^{n_{r+2}}(p_{r+1}{}^{n_{r+1}} + \cdots + 1)) \ldots = 1$$

When A_z and B_z are reduced when multiplied by the reciprocal, if the products excluding the reduced variable are expressed as $A_1 A_2 \ldots A_x$ and $B_1 B_2 \ldots B_x$,

$$A_1 A_2 \ldots A_x p^n(p_x{}^{n_x} + \cdots + 1) = B_1 B_2 \ldots B_x p_x{}^{n_x}(p^n + \cdots + 1)$$

When $n = n_x$ it becomes contradiction like above proof.

When $n_x < n$, a contradiction arises in the case of $p_x \neq p$ since prime number p is not included in the right side. It becomes inconsistent in the case of $p_x = p$ since only the left side includes the prime number p. Therefore, when n is arbitrary, the number of combinations (a, b, p, n) of solutions for $R = 1$ is one or less.

When $(a, b, p, n) = (1, 1, -2, -2)$,

$R = a(p^n + \cdots + 1)/(2bp^n) = (p^{n+1} - 1)/(2(p-1)p^n) = (1/p - 1)/(2(p-1)/p^2)$

$R = p(1-p)/(2(p-1)) = -p/2 = 1$

Substituting $n = p = -2$ into equation (B),

$A_1A_2 \ldots A_x(p-1)p^n({p_x}^{n_x} + \cdots + 1) = B_1B_2 \ldots B_x{p_x}^{n_x}(p^{n+1} - 1)$

$A_1A_2 \ldots A_x(p-1)/p^2({p_x}^{n_x} + \cdots + 1) = B_1B_2 \ldots B_x{p_x}^{n_x}(1/p - 1)$

$A_1A_2 \ldots A_x(p-1)({p_x}^{n_x} + \cdots + 1) = B_1B_2 \ldots B_xp(1-p){p_x}^{n_x}$

$A_1A_2 \ldots A_x({p_x}^{n_x} + \cdots + 1) = -pB_1B_2 \ldots B_x{p_x}^{n_x}$

$A_1A_2 \ldots A_x({p_x}^{n_x} + \cdots + 1) = 2B_1B_2 \ldots B_x{p_x}^{n_x}$

This expression must be satisfied since equation (A) holds at the final point $(a, b, p, n) = (A_1A_2 \ldots A_x, B_1B_2 \ldots B_x, p_x, n_x)$. However it becomes contradiction since $p = 1$ must be established by the definition of p when $(a, b) = (1, 1)$.

Equation (A) holds when $(a, b, p, n) = (\infty, \infty, p, n)$. Since there is at most one solution of odd perfect numbers, there is no solution in the range of $a < \infty, b < \infty$. Therefore, there are no odd perfect numbers.

4. Complement

From equation ⑤,

$2bp^n(p-1) = a(p^{n+1}-1)$

$2 = a(p^{n+1}-1)/(bp^n(p-1))$

$2 = (p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)\dots(p_r{}^{q_r+1}-1)(p^{n+1}-1)$

$/(p_1{}^{q_1}p_2{}^{q_2}\dots p_r{}^{q_r}p^n(p_1-1)(p_2-1)\dots(p_r-1)(p-1))$

$2(p_1{}^{q_1+1}-p_1{}^{q_1})(p_2{}^{q_2+1}-p_2{}^{q_2})\dots(p_r{}^{q_r+1}-p_r{}^{q_r})(p^{n+1}-p^n)$

$= (p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)\dots(p_r{}^{q_r+1}-1)(p^{n+1}-1)$

We consider when $r = 2$.

$(p_1{}^{q_1+1}-1)(p_2{}^{q_2+1}-1)(p^{n+1}-1) = 2(p_1{}^{q_1+1}-p_1{}^{q_1})(p_2{}^{q_2+1}-p_2{}^{q_2})(p^{n+1}-p^n)$

Let s, t, u be integers,

$s = p_1{}^{q_1+1}-1$

$t = p_2{}^{q_2+1}-1$

$u = p^{n+1}-1$

are.

$stu = 2(p_1{}^{q_1+1}-1-(p_1{}^{q_1}-1))(p_2{}^{q_2+1}-1-(p_2{}^{q_2}-1))(p^{n+1}-1-(p^n-1))$

$stu = 2(s-(s+1)/p_1+1)(t-(t+1)/p_2+1)(u-(u+1)/p+1)$

$pp_1p_2stu = 2((s+1)p_1-(s+1))((t+1)p_2+(t+1))\big((u+1)p+(u+1)\big)$

$pp_1p_2stu = 2(s+1)(p_1-1)(t+1)(p_2-1)(u+1)(p-1)$

$stu/((s+1)(t+1)(u+1)) = 2(p_1-1)(p_2-1)(p-1)/(p_1p_2p)$

Since $stu/((s+1)(t+1)(u+1))$ is a monotonically increasing function for variables s, t and u, if

$s \geqq 3^{2+1}-1 = 26,\ p_1 = 3,\ q_1 = 2$

$t \geqq 7^{2+1}-1 = 342,\ p_2 = 7,\ q_2 = 2$

$u \geqq 5^2-1 = 24,\ p = 5,\ n = 1$

holds,

$stu/((s+1)(t+1)(u+1)) \geqq 26\times 342\times 24/(27\times 343\times 25) = 7904/8575$

$2(p_1-1)(p_2-1)(p-1)/(p_1p_2p) = 2\times 2\times 6\times 4/(3\times 7\times 5) = 32/35$

Since $stu/((s+1)(t+1)(u+1))$ is limited to 1 when s, t and u are infinite,
$stu/((s+1)(t+1)(u+1)) < 1$

If $f(p_1, p_2, p) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$ holds, it is sufficient to consider a combination where $f(p_1, p_2, p) < 1$.
$f(3,7,5) = 2 \times 2 \times 6 \times 4/(3 \times 7 \times 5) = 32/35$
$f(3,11,5) = 2 \times 2 \times 10 \times 4/(3 \times 11 \times 5) = 32/33$
$f(3,13,5) = 2 \times 2 \times 12 \times 4/(3 \times 13 \times 5) = 64/65$
$f(3,17,5) = 2 \times 2 \times 16 \times 4/(3 \times 17 \times 5) = 256/255$
$f(3,7,13) = 2 \times 2 \times 6 \times 12/(3 \times 7 \times 13) = 96/91$
$f(3,5,17) = 2 \times 2 \times 4 \times 16/(3 \times 5 \times 17) = 256/255$
From the above, when $r = 2$, a combination $(p_1, p_2, p) = (3,7,5), (3,11,5), (3,13,5)$ can be considered.

Let q_k be 2 and $n = 1$, if $g(p_1, p_2, p) = ({p_1}^3 - 1)({p_2}^3 - 1)(p^2 - 1)/({p_1}^3 {p_2}^3 p^2)$,
$g(3,7,5) = 26 \times 342 \times 24/(3^3 7^3 5^2) = 7904/8575 > 32/35$
$g(3,11,5) = 26 \times 1330 \times 24/(3^3 11^3 5^2) = 55328/59895$
$g(3,13,5) = 26 \times 2196 \times 24/(3^3 13^3 5^2) = 3904/4225$
Since the function g is the minimum in the case of $q_k = 2$ and $n = 1$, there is no solution q_k and n when $g > f$, so the case of $(p_1, p_2, p) = (3,7,5)$ becomes unsuitable.

$$stu/((s+1)(t+1)(u+1)) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$$
$$({p_1}^{q_1+1} - 1)({p_2}^{q_2+1} - 1)(p^{n+1} - 1)/({p_1}^{q_1+1} {p_2}^{q_2+1} p^{n+1})$$
$$= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$$

If $F(p_1, p_2, p) = (p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)$,
$F({p_1}^{q_1+1}, {p_2}^{q_2+1}, p^{n+1}) = 2F(p_1, p_2, p)$

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6. References

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