Proof that there are no odd perfect numbers

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April 30th, 2019
1. Abstract

For $y$ to be a perfect number, if one of the prime factors is $p$, the exponent of $p$ is an integer $n (n \geq 1)$, the prime factors other than $p$ are $p_1, p_2, p_3, \ldots, p_r$, and the even exponent of $p_k$ is $q_k$,

\[ y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k}) / (2p^n) = \prod_{k=1}^{r} p_k^{q_k} \]

must be satisfied. Let $m$ and $q$ be non-negative integers,

\[ n = 4m + 1 \]
\[ p = 4q + 1 \]

Letting $b$ and $c$ be odd integers, satisfying following expressions,

\[ b = \prod_{k=1}^{r} p_k^{q_k} \]
\[ c = \prod_{k=1}^{r} (1 + p_k + p_k^2 + \cdots + p_k^{q_k}) / p^n \]
\[ 2b = c(p^n + \cdots + 1) \]

is established. This is a known content. By the consideration of this research paper, since it turns out that contradiction arises from the $n$ order equation of $p$, we have obtained the conclusion that there are no odd perfect numbers.

2. Introduction

The perfect number is one in which the sum of the divisors other than itself is the same value as itself, and the smallest perfect number is

\[ 1 + 2 + 3 = 6 \]

It is 6. Whether an odd perfect number exists or not is currently an unsolved problem.
3. Proof

An odd perfect number is \( y \), one of them is an odd prime number \( p \), an exponent of \( p \) is an integer \( n \) \((n \geq 1)\). Let \( p_1, p_2, p_3, \ldots, p_r \) be the odd prime numbers of factors other than \( p \), \( q_k \) the index of \( p_k \), and variable \( a \) be the sum of product combinations other than prime \( p \).

\[
a = \prod_{k=1}^{r} (1 + p_k + p_k^2 + \ldots + p_k^{q_k}) \quad \ldots \quad (1)
\]

The number of terms \( N \) of variable \( a \) is

\[
N = \prod_{k=1}^{r} (q_k + 1) \quad \ldots \quad (2)
\]

When \( y \) is a perfect number,

\[
y = a(1 + p + p^2 + \ldots + p^n) - y \quad (n > 0)
\]

is established.

\[
a \sum_{k=0}^{n} p^k /2 = y
\]

\[
a \sum_{k=0}^{n} p^k / (2p^n) = y/p^n \quad \ldots \quad (3)
\]

3.1. If \( q_k \) has at least one odd integer

Letting the number of terms where \( q_k \) is an odd integer be a positive integer \( u \), because \( y/p^n = \prod_{k=1}^{r} p_k^{q_k} \) is an odd integer, the denominator on the left side of expression \( (3) \) has a prime factor 2, from expression \( (2) \) variable \( a \) has more than \( u \) prime factor 2 and variable \( a \) is an even integer. Therefore \( \sum_{k=0}^{n} p^k \) must be an odd integer, \( n \) is an even integer and \( u \) is 1.

3.2. When all \( q_k \) are even integers

\( y/p^n \) is an odd integer, the denominator on the left side of expression \( (3) \) is an even integer, and since \( N \) is and odd integer when \( q_k \) are all even integers, variable \( a \) is and odd integer. Therefore \( \sum_{k=0}^{n} p^k \) is necessary to include one prime factor 2, \( \sum_{k=0}^{n} p^k \equiv 0 \pmod{2} \) is established, and \( n \) must be an odd integer.

From 3.1, 3.2, in order to have an odd perfect number, only one exponent of the prime factor of \( y \) must be an odd integer and variable \( a \) must be an odd integer. We consider the case of 3.2 below.
In order for \( y \) to be a perfect number, the following expression must be established.

\[
y/p^n = (1 + p + p^2 + \cdots + p^n) \prod_{k=1}^{r} \frac{(1 + p_k + p_k^2 + \cdots + p_k^{q_k})}{(2 p_k^n)} = \prod_{k=1}^{r} p_k^{q_k}
\]

However, \( q_1, q_2, \ldots, q_r \) are all even integers.

Here, let \( b \) be an integer

\[
b = \prod_{k=1}^{r} p_k^{q_k} \quad \text{④}
\]

A following expression is established.

\[
y/p^n = a(1 + p + p^2 + \cdots + p^n)/(2p^n) = b
\]

\[
a(p^{n+1} - 1)/(2(p - 1)p^n) = b
\]

\[
(a - 2b)p^{n+1} + 2bp^n - a = 0 \quad \text{⑤}
\]

Because it is an \( n + 1 \) order equation of \( p \), the solution of the odd prime \( p \) is \( n + 1 \) at most.

\[
(ap - 2bp + 2b)p^n = a
\]

Since \( ap - 2bp + 2b \) is an odd integer, \( a/p^n \) is an odd integer, which is \( c \).

\[
ap - 2bp + 2b = c \quad (c > 0) \quad \text{⑥}
\]

\[
(2b - a)p = 2b - c
\]

Since variable \( a \) is an odd integer, \( 2b - a \) is an odd integer and \( 2b - a \neq 0 \).

\[
p = (2b - c)/(2b - a)
\]
Since $n \geq 1$
\[a - c = cp^n - c \geq cp - c > 0\]
a > c
is.

From equation ⑥
\[2b(p - 1) - (ap - c) = 0\]
\[2b - c(p^{n+1} - 1)/(p - 1) = 0\]
\[(p^n + \cdots + 1)/2\] is an odd integer, $n = 4m + 1$ is required with $m$ as an integer.
\[2b(p - 1) = c(p^{n+1} - 1)\]
\[2b = c(p^n + \cdots + 1)\]
\[2b = c(p + 1)(p^{n-1} + p^{n-3} + \cdots + 1) \quad \text{⑦}\]
b is an odd integer when $p + 1$ is not a multiple of 4. It is necessary that $p - 1$ be a multiple of 4. A positive integer is taken as $q$.
$p = 4q + 1$
is established.

When $p > 1$
\[p^n - 1 < p^n\]
\[(p^n - 1)/(p - 1) < p^n/(p - 1)\]
\[p^{n-1} + \cdots + 1 < p^n/(p - 1) \quad \text{⑧}\]

Since $p$ is an odd prime number satisfying $p = 4q + 1$ and $p \geq 5$
\[p^{n-1} + \cdots + 1 < p^n/4\]
\[2b - a = c(p^n + \cdots + 1) - cp^n = c(p^{n-1} + \cdots + 1)\]
\[2b - a < cp^n/4 = a/4\]
\[2b < 5a/4\]
a > $8b/5 \quad \text{⑨}$
Let $a_k$ and $b_k$ be integers and if
$a_k = 1 + p_k + p_k^2 + \cdots + p_k^{q_k}$, $b_k = p_k^{q_k}$,
$a_k - b_k < b_k/(p_k - 1)$
$a_k < b_k p_k /(p_k - 1)$

$$a = \prod_{k=1}^{r} a_k < \prod_{k=1}^{r} b_k p_k/(p_k - 1) = b \prod_{k=1}^{r} p_k/(p_k - 1)$$
$a/b < \prod_{k=1}^{r} p_k/(p_k - 1)$

When $r = 1$, since $a/b < 3/2$ is established, it becomes inappropriate contrary to inequality ⑨.

From $a = cp^n$ and expression ⑦,
$2bp^n = a(p^n + \cdots + 1)$
$a(p^n + \cdots + 1)/(2bp^n) = 1 \quad \cdots (A)$

When $r = 1$,
$a = (p_1^{q_1+1} - 1)/(p_1 - 1)$
$b = p_1^{q_1}$
Equation (A) does not hold since there is no odd perfect number when $r = 1$.

Let $R$ be a rational number,
$R = a(p^n + \cdots + 1)/(2bp^n)$
Let $b'$ be a rational number and let $A$ and $B$ to be an integer,
$b' = (p_t^{q_t+1} - 1)/(p_t^{q_t}(p_t - 1)) > 1$
$A = (p_t^{q_t+1} - 1)/(p_t - 1)$
$B = p_t^{q_t}$
Multiplying \( R \) by \( b' \), there are both cases that \( p_r \) increases \( p \) or does not change. When multiplied by \( b' \), the rate of change of \( R \) is
\[
\frac{Ap^n(p^n + \cdots + 1)}{Bp'^n(p^n + \cdots + 1)},
\]
if \( p \) after variation is \( p' \). If the rate of change of \( R \) is 1,
\[
\frac{Ap^n(p^n + \cdots + 1)}{Bp'^n(p^n + \cdots + 1)} = 1
\]
This expression does not hold, since the right side is not a multiple of \( p \) when \( p' > p \), and \( A > B \) holds when \( p' = p \). Due to this operation, \( R \) may be larger or smaller than the original value, since the rate of change of \( R \) does not become 1.

From \( R \neq 1 \) and \( a = cp^n \) for some \( r \), also multiplying fractions \( b' = A_1/B_1 \),
\[
b'' = A_2/B_2, \ldots b''' = A_x/B_x,
\]
if \( R = 1 \) holds finally,
\[
a/(2b) \times A_1/B_1 \times A_2/B_2 \ldots A_x(p_x^n + \cdots + 1)/(B_xp_x^n(p_x^n + \cdots + 1)) = 1
\]
When \( p_x > p \), it becomes inconsistent since the right side of this expression does not include \( p \) as a factor.

When \( p_x = p \),
\[
cp^n(p^n + \cdots + 1)A_1A_2 \ldots A_x = c(p^n + \cdots + 1)p^n
\]
\( A_1A_2 \ldots A_x = 1 \)
It becomes contradiction, since this expression is not established. Therefore, \( a = cp^n \) holds at one point where \( R = 1 \).
Assuming that \( R = 1 \) in some \( r \) by multiplying fractions \( b' = A_1/B_1 \), \( b'' = A_2/B_2 \), \( \ldots \) \( b'''' = A_x/B_x \), if \( R = 1 \) holds,
\[
1 \times A_1 p^n (p_1^n + \cdots + 1)/(B_1 p_1^n (p^n + \cdots + 1)) \times A_2 p_x^n (p_2^n + \cdots + 1)/(B_2 p_2^n (p_1^n + \cdots + 1)) \times A_3 p_{x-1}^n (p_x^n + \cdots + 1)/(B_3 p_{x-1}^n (p_{x-1}^n + \cdots + 1)) = 1
\]
\( A_1 A_2 \ldots A_x p^n (p_1^n + \cdots + 1) = B_1 B_2 \ldots B_x p_{x}^n (p^n + \cdots + 1) \)

When \( p_x > p \), it becomes inconsistent since the right side of this expression does not include \( p \) as a factor.

When \( p_x = p \),
\( A_1 A_2 \ldots A_x = B_1 B_2 \ldots B_x \)
is established. It becomes contradiction. Therefore, when \( n \) is fixed, the number of values of \( r \) for which \( R = 1 \) is one or less.

Assuming that \( R = 1 \) in some \( r \) by multiplying fractions \( b' = A_1/B_1 \), \( b'' = A_2/B_2 \), \( \ldots \) \( b'''' = A_x/B_x \) and reciprocal of fraction previously multiplied, if \( R = 1 \) holds. At this time, assuming that \( n \) also changes, the change rate when multiplying by \( A_1/B_1 \) is
\[
A_1 p^n (p_1^n + \cdots + 1)/(B_1 p_1^n (p^n + \cdots + 1))
\]
The rate of change when multiplying \( p_{r+2}^{r+2} / (p_{r+2}^{r+2} + \cdots + 1) \) after this is
\[
p_1 q_1 p_{r+1}^{r+1} (p_{r+2}^{r+2} + \cdots + 1) / ((p_1 q_1 + \cdots + 1) p_{r+2}^{r+2} (p_{r+1}^{r+1} + \cdots + 1))
\]
\[
1 \times A_1 p^n (p_{r+1}^{r+1} + \cdots + 1)/(B_1 p_{r+1}^{r+1} (p^n + \cdots + 1)) \times p_1 q_1 p_{r+1}^{r+1} (p_{r+2}^{r+2} + \cdots + 1)/(p_1 q_1 + \cdots + 1) p_{r+2}^{r+2} (p_{r+1}^{r+1} + \cdots + 1)) \ldots = 1
\]
When \( A_2 \) and \( B_2 \) are reduced when multiplied by the reciprocal, if the products excluding the reduced variable are expressed as \( A_1 A_2 \ldots A_x \) and \( B_1 B_2 \ldots B_x \),
\( A_1 A_2 \ldots A_x p^n (p_1^n + \cdots + 1) = B_1 B_2 \ldots B_x p_{x}^n (p^n + \cdots + 1) \)

Since it becomes contradiction like above proof, when \( n \) is arbitrary, the number of combinations \((a,b,p,n)\) of solutions with \( R = 1 \) is 1 or less.
Assuming that \( R = 1 \) in some \( r \) and by multiplying \( R = 1 \) holds by the reciprocal of the fraction multiplied until then. At this time, it is assumed that \( n \) changes, and the changed \( p \) is \( p'_x \), and the change rate when multiplied by \( p_r^{qr}/(p_r^{qr} + \ldots + 1) \) is

\[
p_r^a p^{n(p'_n - 1^{n-1} + \ldots + 1)} / ((p_r^{qr} + \ldots + 1)p'_n^{-1} n^{-1}(p^n + \ldots + 1))
\]

Since when \( r = 0 \) and \( n_0 = 1 \), \( R = 1 \), \( a = 1 \), \( b = 1 \) and \( p'_0 = 1 \) hold finally,

\[
1 \times p_r^{qr} p^{n(p'_n - 1^{n-1} + \ldots + 1)} / ((p_r^{qr} + \ldots + 1)p'_n^{-1} n^{-1}(p^n + \ldots + 1))
\]

\[
\times p_{r-1}^{qr-1} p'_n^{-1} n^{-1}(p'_{n-2}^{n-2} + \ldots + 1) / ((p_{r-1}^{qr-1} + \ldots + 1)p'_n^{n-2} n^{-2}(p'_n - 1^{n-1} + \ldots + 1)) ...
\]

\[
p^n(p'_0 n_0 + \ldots + 1)a / ((p^n + \ldots + 1)p'_0 n_0 b) = 1
\]

\[
p^n = p^n + \ldots + 1
\]

\[
p^{n-1} + \ldots + 1 = 0
\]

It becomes contradiction since there are no real solution where \( p > 0 \). From the above, there are no odd perfect numbers.
4. Complement

From equation ⑤,
\[ \begin{align*}
2bp^n(p-1) &= a(p^{n+1} - 1) \\
2 &= a(p^{n+1} - 1)/(bp^n(p-1)) \\
2 &= (p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1) \cdots (p_r^{q_r+1} - 1)(p^{n+1} - 1) \\
&\quad / (p_1^{q_1}p_2^{q_2} \cdots p_r^{q_r} + p(p_1 - 1)(p_2 - 1) \cdots (p_r - 1)(p - 1)) \\
2(p_1^{q_1+1} - p_1^{q_1})(p_2^{q_2+1} - p_2^{q_2}) \cdots (p_r^{q_r+1} - p_r^{q_r})(p^{n+1} - p^n)
\end{align*} \]

We consider when \( r = 2 \).

\[ (p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1)(p^{n+1} - 1) = 2(p_1^{q_1+1} - p_1^{q_1})(p_2^{q_2+1} - p_2^{q_2})(p^{n+1} - p^n) \]

Let \( s, t, u \) be integers,

\[ \begin{align*}
s &= p_1^{q_1+1} - 1 \\
t &= p_2^{q_2+1} - 1 \\
u &= p^{n+1} - 1
\end{align*} \]

are.

\[ \begin{align*}
stu &= 2(p_1^{q_1+1} - 1 - (p_1^{q_1} - 1))(p_2^{q_2+1} - 1 - (p_2^{q_2} - 1))(p^{n+1} - 1 - (p^n - 1)) \\
stu &= 2(s - (s + 1)/p_1 + 1)(t - (t + 1)/p_2 + 1)(u - (u + 1)/p + 1) \\
pp_1p_2stu &= 2((s + 1)p_1 - (s + 1))(t + 1)p_2 + (t + 1))((u + 1)p + (u + 1)) \\
pp_1p_2stu &= 2(s + 1)(p_1 - 1)(p_2 - 1)(u + 1)(p - 1) \\
stu/((s + 1)(t + 1)(u + 1)) &= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p)
\end{align*} \]

Since \( stu/((s + 1)(t + 1)(u + 1)) \) is a monotonically increasing function for variables \( s, t \) and \( u \), if

\[ \begin{align*}
\begin{align*}
&\begin{align*}
s \geq 3^{2^{n+1} - 1} - 1 = 26, & p_1 = 3, & q_1 = 2 \\
t \geq 7^{2^{n+1} - 1} - 1 = 342, & p_2 = 7, & q_2 = 2 \\
u \geq 5^{2^2 - 1} - 1 = 24, & p = 5, & n = 1
\end{align*}
\end{align*}
\end{align*} \]

holds,

\[ \begin{align*}
stu/((s + 1)(t + 1)(u + 1)) &\geq 26 \times 342 \times 24 / (27 \times 343 \times 25) = 7904 / 8575 \\
2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1p_2p) &\approx 2 \times 2 \times 6 \times 4 / (3 \times 7 \times 5) = 32 / 35
\end{align*} \]
Since \(stu/((s + 1)(t + 1)(u + 1))\) is limited to 1 when \(s, t\) and \(u\) are infinite, 
\[stu/((s + 1)(t + 1)(u + 1)) < 1\]

If \(f(p_1, p_2, p) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)\) holds, it is sufficient to consider a combination where \(f(p_1, p_2, p) < 1\).
\[
of(3, 7, 5) = 2 \times 2 \times 6 \times 4 / (3 \times 7 \times 5) = 32/35
\]
\[
of(3, 11, 5) = 2 \times 2 \times 10 \times 4 / (3 \times 11 \times 5) = 32/33
\]
\[
of(3, 13, 5) = 2 \times 2 \times 12 \times 4 / (3 \times 13 \times 5) = 64/65
\]
\[
of(3, 17, 5) = 2 \times 2 \times 16 \times 4 / (3 \times 17 \times 5) = 256/255
\]
\[
of(3, 7, 13) = 2 \times 2 \times 6 \times 12 / (3 \times 7 \times 13) = 96/91
\]
\[
of(3, 5, 17) = 2 \times 2 \times 4 \times 16 / (3 \times 5 \times 17) = 256/255
\]
From the above, when \(r = 2\), a combination \((p_1, p_2, p) = (3, 7, 5), (3, 11, 5), (3, 13, 5)\) can be considered.

Let \(q_k\) be 2 and \(n = 1\), if \(g(p_1, p_2, p) = (p_1^2 - 1)(p_2^2 - 1)(p^2 - 1)/(p_1^3 p_2^3 p^2)\),
\[
g(3, 7, 5) = 26 \times 342 \times 24 / (3^3 7^3 5^2) = 7904/8575 > 32/35
\]
\[
g(3, 11, 5) = 26 \times 1330 \times 24 / (3^3 11^3 5^2) = 55328/59895
\]
\[
g(3, 13, 5) = 26 \times 2196 \times 24 / (3^3 13^3 5^2) = 3904/4225
\]
Since the function \(g\) is the minimum in the case of \(q_k = 2\) and \(n = 1\), there is no solution \(q_k\) and \(n\) when \(g > f\), so the case of \((p_1, p_2, p) = (3, 7, 5)\) becomes unsuitable.

\[
stu/((s + 1)(t + 1)(u + 1)) = 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)
\]
\[
(p_1^{q_1+1} - 1)(p_2^{q_2+1} - 1)(p^{n+1} - 1)/(p_1^{q_1+1} p_2^{q_2+1} p^{n+1})
\]
\[
= 2(p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)
\]

If \(F(p_1, p_2, p) = (p_1 - 1)(p_2 - 1)(p - 1)/(p_1 p_2 p)\),
\[
F(p_1^{q_1+1}, p_2^{q_2+1}, p^{n+1}) = 2F(p_1, p_2, p)
\]
5. Acknowledgement

In writing this research document, we asked anonymous reviewers to point out several tens of mistakes. We would like to thank you for giving appropriate guidance and counter-arguments.

6. References

Hiroyuki Kojima "The world is made of prime numbers" Kadokawa Shoten, 2017
Fumio Sairaiji·Kenichi Shimizu "A story that prime is playing" Kodansha, 2015