On the Mathematical Formulation of 4D Fock Space
Paul J. Werbos, April 27, 2019

Abstract

Ordinary Fock space, defined as concatenated combination of three-dimensional spaces is the main workhorse of the canonical version of quantum field theory. It is also fundamental to the modern version of quantum field theory without observers, as in the theory of Everett, Wheeler and Deutsch, which led to the theory of the universal quantum computer. It also has many practical uses, from quantum optics to the closure of turbulence and the derivation of the emergent behavior of space-time dynamical systems. But many of us believe that time and space are interchangeable to some degree, and that methods based on 4D Fock space are needed. This letter describes two simple mathematical tools to help make that possible. It discusses how computing the emergent statistics of such systems is a proper generalization of value function estimation in machine learning, which may therefore carry the same complexities with it.

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1. The Challenge: Why We Might Need More Than 3D Fock Space

The canonical forms of quantum field theory (QFT) today are based on the Schrodinger equation:

$$\dot{\Psi} = iH\Psi \quad , \qquad (1)$$

where the wave function Ψ(z*) is defined over points z in three-dimensional Fock space [1,2]. Three dimensional (3D) Fock Space is not a three dimensional space; it is the vector sum of spaces of continuous dimension 3N, for all N from zero to infinity, with discrete “spin” dimensions added as well for each group of 3 dimensions. It is essentially the modern version of what Von Neumann called “configuration space.”

Intuitively, it is hard to believe that the true, ultimate laws of quantum field theory are based on equation (1), because equation (1) treats time as something different from space, even though Lorentz invariance is one of the most well-established laws of physics. For that reason, mathematicians trying to reformulate quantum field theory without ejecting the use of Fock space altogether have tried to reformulate it in terms of 4D Fock space, involving wave functions Ψ(z) where z refers to a point in 4D Fock space [3]. Unfortunately, their efforts did not meet the standards of rigor of mathematics any more than traditional canonical QFT did, because they relied heavily on quantities such as the square norm of particle world-lines across space time for which the square norm would naturally be infinite. In order to construct a mathematically well-defined and plausible formulation of quantum realism [2], we need to find a way to fix these details.

Furthermore, Fock space is important today as a mathematical tool for describing the emergent behavior implied by models over ordinary Minkowski space, of which there are a great variety. High energy physics tends to make strong implicit assumptions based on common sense about the equivalence between Feynmann path ideas and correlation functions which are actually defined over 4D Fock space [1]; it would be good to have a mathematical formulation as needed to prove those assumptions, or to fix them to make them more precise. Likewise, more axiomatic formulations of QFT over 3+1 dimensions [4] have statistical implications over space and time better represented in terms of 4D Fock space than 3D.

What drove me to submit this note today is another consideration altogether, another area where 4D Fock space is important. It concerns "the closure of turbulence," the effort to compute the emergent statistics of dynamical systems governed either by [1] or by nonlinear partial differential equations. For [1], it is well-known that the possible equilibrium ensembles are given by the quantum grand canonical density matrix [5], the QFT version of a Boltzmann distribution, which is effectively a probability distribution for possible values of the wave function at any time. But what about correlations ***across time*** ? How can we represent allowed equilibria of the whole system of "correlations" (actually, covariations) across Minkowski space-time?

Furthermore, practical and empirical quantum optics [6-11] depends heavily on the "coherence representation" of light (not to be confused with the wave function), which uses "distribution functions" to map between classical representations and (simplified) 3D Fock space. I have generalized the standard "P" distribution function so as to map back and forth between classical fields in general and the full 3D Fock space proper [12-14], but what is the full 4D version?

When the usual Boltzmann type equilibrium tools are applied to nonlinear PDE, I have derived a similar result, or entropy function, for a large class of PDE . Under my generalized P mapping, this implies an equilibrium density matrix over the 3D Fock space at the end of that mapping [14]. But again, what of the covariations across space-time?

2. How to Meet the Challenge

2.1. Goals of This letter

This letter will not fully resolve all the complex, connected challenges mentioned in the previous section. However, it will suggest a few simple, basic mathematical tools needed to make solutions possible. In an ideal world, I would take years to develop each of the applications in some detail, and work their way through the journals of highest impact factor, but after retirement and after age 70 one cannot be sure of having enough time left to do so.

There are two main tools required here, the 4D version of the generalized P mapping and an “interest rate” regularization. There are analogies between the interest rate regularization and tools necessary to the analysis of the Bellman equation and others used in Reinforcement Learning and Approximate Dynamic Programming (RLADP [15]). There may also be a more distant analogy to the breakthroughs of Von Neumann in quantum mechanics, in working with operators which did not fit earlier notions of finite norm and existence of an adjoint.

2.2 The 4D generalized P mapping

In [14], the generalized 3D P mapping is defined as:

$$\left|S\right\rangle = Ze^{\sum_j \int \alpha_j(\underline{p}) a_j^H(\underline{p}) d^3\underline{p}} \left|0\right\rangle, \qquad (2)$$

where $\alpha_j(\underline{x})$ for j = 1 to m is a set of m functions or field over points $\underline{x}$ in 3D space, where a^H_j is the standard creation operator for 3D Fock space [1,12], where |0> is the vacuum state of 3D Fock-Hilbert space, where |S> is the corresponding wave function considered as a state |S> over 3D Fock-Hilbert space, and where Z is a scalar term used to normalize the length of |S> to 1. As in canonical QFT (1), we use $\underline{p}$ to denote Fourier transforms from $\underline{x}$. An ensemble of possible classical states is of course mapped into the density matrix ρ, the sum of $p_k|S_k><S_k|$.

Conceptually, the 4D generalized mapping is exactly the same in appearance, except that we replace $\underline{p}$ by ***x***, where ***x*** is a point in Minkowski space, and the integration is over all four dimensions. Classical-quantum equivalence is actually easier to obtain in the 4D representation than in the 3D representation, if we consider the S matrix for scattering as a mapping from an incoming statistical ensemble to an outgoing ensemble. In the 3D case, a function of p appears in the denominator of the definition of the field operators, but that is not necessary in the 4D case. This is discussed in a paper 30 years old in my files, but I have not pursued it more because physics today is better prepared for improvements in the 3D formulations.

2.3 Interest Rate Regularization

There are many, many different types of regularization used in physics, in engineering and elsewhere. Regularization is often used in conjunction with mass and charge renormalization, in a way which changes the very definition of QFTs. Here, however, it can be used in a more passive way, to analyze equilibrium statistics.

The density matrix ρ_4 over 4D Fock space is made up of elements which may be written as:

$\rho_4(\boldsymbol{x}_1, s_1, \boldsymbol{x}_2, s_2, \ldots \boldsymbol{x}_n, s_n ; \boldsymbol{x}'_1, s'_1, \boldsymbol{x}'_2, s'_2, \ldots \boldsymbol{x}'_m, s'_m)$

where the ***x*** coordinates are vectors in Minkowski space and the discrete s coordinates are spin coordinates [1,3]. For convenience, let us define the array ***v*** of length n+m as the array of the n 4-vectors ***x*** concatenated with the m 4-vectors ***x'***. Let us define the raw (unnormalized) regularized density matrix $\rho^*_{4,0}(\beta)$ as the corresponding matrix made up of elements:

$$\rho_4(\boldsymbol{x}_1, s_1, \boldsymbol{x}_2, s_2,\ldots \boldsymbol{x}_n, s_n ; \boldsymbol{x}'_1, s'_1, \boldsymbol{x}'_2, s'_2,\ldots \boldsymbol{x}'_m, s'_m) \, \Pi\{i,j\} \exp(-\beta\Delta(v_i - v_j)) \qquad (3)$$

where Δ is a simple distance function such as the L1 norm (the sum of the absolute value of the differences along all four coordinates) or even just the absolute difference in the time coordinate. When the difference in time is used, this is very similar to the use of interest rates (aka discount factors) in economics and intelligent systems [15-18].

The full regularized density matrix is defined as

$$\rho^*_4(\beta) = Z \, \rho^*_{4,0}(\beta) \qquad (4),$$

where Z is just the scalar which normalizes the length of the matrix to one.

For many (almost all?) relevant field theories, the matrix $\rho^*_4(\beta)$ may be well defined for all values of β between zero (not including zero) and some small positive value. The limit as β goes to zero may also be well-defined. That limit is what we want to study, when we are studying the emergent complex covariations which result from that field theory.

2.4 Analogies and Possibilities

Many years ago, mathematicians were startled to learn just how diverse and tricky the emergent behavior can be from diverse types of simple one-dimensional systems [19]. There are many indications that systems over Minkowski space or Fock space may have even more diversity, beyond what we begin to understand as yet. For some systems, equation 3 may be necessary and sufficient. For some systems, it may be unnecessary to take a limit in order to derive ρ_4. But for others, the limit may not be well defined.

The calculation of ρ_4 has a strong analogy to the calculation of the value function J in RLADP [16,17]. In fact, the J estimate is essentially just an estimate of the first statistical moment (the mean) of one selected variable (U) in a one-dimensional dynamical system. That is a narrow special aspect of the more general calculation we are describing here. The more general system may actually have uses in computer technology as well as physics, because anticipation of the fate of more than one variable is often of interest to system developers.

In RLADP, it is always possible to get finite J estimates when the interest rate is more than zero (for normal types of deterministic or stochastic dynamic system), but there are cases where the limit as the interest rate (r) goes to zero does not exist or is difficult to compute. Even when the limit exists, it often works best to start from a finite value of r and lower it slowly, so as to gradually expand the degree of vision or foresight of a system. There are times when greater foresight reveals that a system is in a kind of crossroads situation [17], which basically rules out the use of more simplistic decision systems and calls for certain modifications [16,17], which are generalizations of earlier concepts by Ron Howard from iterative dynamic programming. We do not know yet what kinds of PDE or stochastic ODE or QFTs would have that kind of property, and what it would imply.

It should also be noted that the Lagrange-Euler equations assumed widely in physics are a very special class of PDE. They are essentially the special case over space time of the equations of RLADP without stochastic disturbance. This raises an interesting question: to what extent do those equations actually predict the complex, tricky kind of behavior which one might expect from an intelligent (RLADP) certain?

3. Some concluding thoughts

The use of methods over 3D Fock space may be thought of as drinking from a firehose, analyzing a very complex changing matrix of information. Analysis over 4D Fock space is more like drinking from an ocean, a vast system over space and time. Both types of analysis have their uses. Availability of both types of method, and of more classical methods, may provide a richer integrated understanding of many of the dynamical systems of importance to us.

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