Refutation of Ishihara's tricks and (seemingly) impossible theorems in constructive mathematics

Abstract: The precise definition of LPO and Ishihara's tricks as rendered in four equations are *not* tautologous. This refutes LPO and Ishihara's tricks. What follows is that (seemingly) impossible theorems in constructive mathematics are denied as theorems. Therefore those conjectures are *non* tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with `T`autology as the designated proof value, **`F`** as contradiction, `N` as truthity (non-contingency), and `C` as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

LET ~ Not, ¬ ; + Or, ∨, ∪, ⊔ ; - Not Or; & And, ∧, ∩, ⊓, ·; \ Not And;
\> Imply, greater than, →, ⇒, ↦, ≻, ⊃, ↠ ;
< Not Imply, less than, ∈, ≺, ⊂, ⊬, ⊭, ↞, ≲ ;
= Equivalent, ≡, :=, ⇐⇒, ↔, ≜, ≈, ≃ ; @ Not Equivalent, ≠;
% possibility, for one or some, ∃, ◊, M; # necessity, for every or all, ∀, □, L;
(z=z) `T` as tautology, ⊤, ordinal 3; (z@z) **`F`** as contradiction, Ø, Null, ⊥ , zero;
(%z>#z) `N` as non-contingency, Δ, ordinal 1;
(%z<#z) `C` as contingency, ∇, ordinal 2;
~(y < x) (x ≤ y), (x ⊆ y); (A=B) (A~B); (B>A) (A⊢B); (B>A) (A⊨B).
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: Diener, H.; Hendtlass, M. (2019).
(Seemingly) impossible theorems in constructive mathematics. arxiv.org/pdf/1904.11378.pdf
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We make frequent reference to two classically valid "omniscience principles", the limited principle of omniscience (LPO) [equivalent to the existence of strongly existensional discontinuous function f] and the weak limited principle of omniscience (WLPO) [equivalent to the existence of discontinuous function f].

The following lemma is very much folklore, at least the WLPO part, however we were unable to find it in the literature. … Here and in the following a function $f: X \to Y$ between two metric (X,σ) and (Y,ρ) is called *strongly extensional* if $\forall x,y \in X: f(x) \neq f(y) \Rightarrow x \neq y$, or to be more precise, $\forall x,y \in X: \rho(f(x), f(y)) > 0 \Rightarrow \sigma(x,y) > 0$. (2.3.1)

((r&((p&x)&(p&y)))>(p@p))>((s&(x&y))>(p@p)) ; "or to be more precise"
`TTTT TTTT TTTT TTTT(16),`
`TTTT TTTT` **`FFFF F`**`T`**`F`**`T(16)` (2.3.2)

Proposition 3 (Ishihara's first trick). *For all positive reals* $\alpha < \beta$*,*
$\exists n \in \mathrm{N}: \rho(f(xn), f(x)) > \alpha \vee \forall n \in \mathrm{N}: \rho\, f(xn), f(x)) < \beta$. (3.1.1)

Remark 3.1.1: The expression *for all positive reals* $\alpha < \beta$ is mapped as (3.1.1.1)
$\alpha<\beta$ and $\alpha*\beta>0$ to #((p<q)&((p&q)>(p@p)))=(p=p) for an antecedent
result of **F**N**FF** **F**N**FF** **F**N**FF** **F**N**FF**. (3.1.1.2)

#((p<q)&((p&q)>(p@p)))>(((%u<v)>((r&((s&(x&u))&(s&x)))<#q))+
((#u<v)>((r&((s&(x&u))&(s&x)))<#p)));

```
TTTT TTTT TTTT TTTT(2),
TCTT TCTT TCTT TCTT(2),
TTTT TTTT TTTT TTTT(4)
```
(3.1.2)

Proposition 4 (Ishihara's second trick). *For all positive reals* $\alpha < \beta$*,*
either we have $\rho(f(xn), f(x)) < \beta$ *eventually, or* $\rho(f(xn), f(x)) > \alpha$ *infinitely often.* (4.1.1)

Remark 4.1.1: The terms *eventually* and *infinitely often* are mapped respectively as possibly and necessarily.

#((p<q)&((p&q)>(p@p)))>(%((r&((s&(x&u))&(s&x)))<#q)+
(#(r&((s&(x&u))&(s&x)))<#p)) ;

```
TCTT TCTT TCTT TCTT(2),
TCTT TCTT TCTT TTTT(2)
```
(4.1.2)

There are various results that improve, generalise, or modify Ishihara's tricks ...

The precise definition of LPO and Ishihara's tricks as rendered in four equations are *not* tautologous. This refutes LPO and Ishihara's tricks. What follows is that (seemingly) impossible theorems in constructive mathematics are denied as theorems.