Refutation of Ishihara’s tricks and (seemingly) impossible theorems in constructive mathematics

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Abstract: The precise definition of LPO and Ishihara’s tricks as rendered in four equations are not tautologous. This refutes LPO and Ishihara’s tricks. What follows is that (seemingly) impossible theorems in constructive mathematics are denied as theorems. Therefore those conjectures are non tautologous fragment of the universal logic VŁ4.

We assume the method and apparatus of Meth8/VŁ4 with Tautology as the designated proof value, F as contradiction, N as truthty (non-contingency), and C as falsity (contingency). The 16-valued truth table is row-major and horizontal, or repeating fragments of 128-tables, sometimes with table counts, for more variables. (See ersatz-systems.com.)

\[ \text{LET} \quad \neg \text{Not, } ; \quad \lor \text{ Or, } , \cup ; \quad \land \text{ And, } , \cap ; \quad \not\lor \text{ Not Or; } \quad \land \text{ And, } , \cap \; ; \quad \not\land \text{ Not And; } \]
\[ > \text{ Imply, greater than, } \Rightarrow , \rightarrow , > , \supset , * ; \]
\[ < \text{ Not Imply, less than, } \Leftarrow , < , \subset , \prec , \preceq ; \quad = \text{ Equivalent, } = , \equiv , := , \leftrightarrow , \Leftarrow ; \quad @ \text{ Not Equivalent, } ; \]
\[ \% \text{ possibility, for one or some, } \exists , \diamond , M ; \quad \# \text{ necessity, for every or all, } \forall , \square , L ; \]
\[ (z=z) \quad \top \text{ as tautology, } , \text{ ordinal } 3 ; \quad (z@z) \quad \bot \text{ as contradiction, } , \text{ Null, } , , \perp ; \]
\[ (\%z>\#z) \quad \Delta \text{ as non-contingency, } , \text{ ordinal } 1 ; \]
\[ (\%z<\#z) \quad \nabla \text{ as contingency, } , \text{ ordinal } 2 ; \]
\[ \neg( y < x) \quad ( x \leq y) ; \quad ( A=B) \quad (A~B) ; \quad (B>A) \quad (A B) ; \quad (B>A) \quad (A B) . \]
Note for clarity, we usually distribute quantifiers onto each designated variable.

From: Diener, H.; Hendtlass, M. (2019). (Seemingly) impossible theorems in constructive mathematics. arxiv.org/pdf/1904.11378.pdf hannes.diener@canterbury.ac.nz, matthew.hendtlass@canterbury.ac.nz

We make frequent reference to two classically valid “omniscience principles”, the limited principle of omniscience (LPO) [equivalent to the existence of strongly existensional discontinuous function f] and the weak limited principle of omniscience (WLPO) [equivalent to the existence of discontinuous function f].

The following lemma is very much folklore, at least the WLPO part, however we were unable to find it in the literature. … Here and in the following a function \( f: X \rightarrow Y \) between two metric \((X, \sigma)\) and \((Y, \rho)\) is called strongly existensional if \( \forall x, y \in X: f(x) \neq f(y) \rightarrow x \neq y \), or to be more precise, \( \forall x, y \in X: \rho(f(x), f(y)) > 0 \Rightarrow \sigma(x, y) > 0 \).

\[ (r\&(p\&x)\&(p\&y))>(p@p))>(s\&(x\&y))>(p@p)) ; "or to be more precise" \]
\[ \text{T T T T} \quad \text{T T T T T T T T} , \quad \text{T T T T} \quad \text{FFFF F F T T F F F F} \]

Proposition 3 (Ishihara’s first trick). For all positive reals \( \alpha < \beta \), \( \exists n \in N: \rho(f(xn), f(x)) > \alpha \land \forall n \in N: \rho(f(xn), f(x)) < \beta \).

\[ \text{T T T T} \quad \text{T T T T} \quad \text{FFT FFT} \quad \text{T T T T (16)} \quad \text{T T T T} \quad \text{FFFF F F T T (16)} \]

(3.1.1)
Remark 3.1.1: The expression for all positive reals $\alpha < \beta$ is mapped as $\alpha < \beta$ and $\alpha \ast \beta > 0$ to $(p \prec q) \&(p \& q) > (p \oplus p)) = (p = p)$ for an antecedent result of $\text{FF \hfill FNFF \hfill FNFF \hfill FNFF}$. (3.1.1.1)

$$\#((p \prec q) \&(p \& q) > (p \oplus p)) = ((p \prec q) \&(p \& q) > (p \oplus p)) = (p = p)$$

Proposition 4 (Ishihara’s second trick). For all positive reals $\alpha < \beta$, either we have $p(f(x_n), f(x)) < \beta$ eventually, or $p(f(x_n), f(x)) > \alpha$ infinitely often. (4.1.1)

Remark 4.1.1: The terms eventually and infinitely often are mapped respectively as possibly and necessarily.

$$\#((p \prec q) \&(p \& q) > (p \oplus p)) = ((p \& (s \& (x \& u)) \& (s \& x)) < \# q)) + ((p \& (s \& (x \& u)) \& (s \& x)) < \# p)) = (p = p)$$

There are various results that improve, generalise, or modify Ishihara’s tricks ...

The precise definition of LPO and Ishihara’s tricks as rendered in four equations are not tautologous. This refutes LPO and Ishihara’s tricks. What follows is that (seemingly) impossible theorems in constructive mathematics are denied as theorems.