Remarks on Infinitesimal Amount of Riemann Zeta Zeros

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Abstract

This remarks proves, that Riemann zeta function has infinitesimal amount of zeros.

Riemann zeta function expressed as follow

$$\zeta(r, \theta) = \sum_{n=1}^{\infty} \frac{1}{n^r \cos \theta + i r \sin \theta}$$

According Abel’s summation formula [1]

$$\zeta(r, \theta) = \sum_{n=1}^{\infty} \frac{1}{n^r \cos \theta + i r \sin \theta} = \lim_{x \to \infty} \frac{x}{x^r \cos \theta + i r \sin \theta} - \int_{u=1}^{\infty} \frac{|u| \, du}{u^r \cos \theta + i r \sin \theta + 1}$$

For $$r > 1, \cos \theta > 0, \sin \theta > 0$$ the first term in (2) goes to zero and we can rewrite (2) without losing of generalisation as follow

$$\zeta(r, \theta) = \frac{\gamma_r \cos \theta + i r \sin \theta}{\gamma_r \cos \theta + i r \sin \theta} - \int_{u=1}^{\infty} \frac{du}{u^r \cos \theta + i r \sin \theta + 1}$$

where $$\gamma_r \cos \theta + i r \sin \theta$$ is constant for each $$r$$ and $$\theta$$. After integration we obtains

$$\zeta(r, \theta) = \frac{\gamma_r \cos \theta + i r \sin \theta - i \frac{\arctan \sin \theta - \cos \theta}{\gamma_r \cos \theta + i r \sin \theta + 1}}{\gamma_r \cos \theta + i r \sin \theta + 1}$$

Let’s restrict $$r$$ and $$\theta$$ to $$r \cos 1 > 0, r \sin \theta > 0$$. So,

$$\zeta(r, \theta) = \frac{\gamma_r \cos \theta + i r \sin \theta + 1}{-r \cos \theta - r i \sin \theta + 1}$$

or

$$\zeta(r, \theta) = \left( (\Re \zeta(r, \theta))^2 + (\Im \zeta(r, \theta))^2 \right) \exp \left( i \arctan \frac{\Im \zeta(r, \theta)}{\Re \zeta(r, \theta)} + i 2 \pi n \right)$$

where

$$\Re \zeta(r, \theta) = \Re \gamma_r \cos \theta + i r \sin \theta + \frac{-r \cos \theta + 1}{(1 - r \cos \theta)^2 + r^2 \sin^2 \theta}$$

$$\Im \zeta(r, \theta) = \Im \gamma_r \cos \theta + i r \sin \theta + \frac{r \sin \theta}{(1 - r \cos \theta)^2 + r^2 \sin^2 \theta}$$
Obviously, for $r >> 1$ we can find infinitesimal amount of Riemann zeta zeros by solving equation as follow

$$\frac{r^2 \Re \gamma r \cos \theta + ir \sin \theta + r \sin \theta}{r^2 \Re \gamma r \cos \theta + ir \sin \theta - r \cos \theta + 1} = 2\pi n = 0, \forall n \in (-\infty, \infty) \quad (9)$$

It implies that Riemann zeta function has infinitesimal amount of zeros.

References